Representing $\Omega_{(l\infty)}$ for real abelian fields

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For real subfields K of a cyclotomic fields $\mathbb{Q}(\zeta)$ we remove the tameness assumption at a given odd prime number l, which was needed in [RW2] in order to establish the equivalence of the Lifted Root Number Conjecture at l and an equivariant main conjecture of Iwasawa theory for abelian extensions of totally real number fields K/k [a].

[a] added in proof: the latter conjecture has meanwhile been confirmed for real $K \subset \mathbb{Q}(\zeta)$, see our paper [manuscripta math. 109 (2002),131-146]

Let K/k be an extension of real number fields which is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$, ζ_n a root of unity of order n, G the Galois group of K/k, and let l be an odd prime number. When l is tamely ramified in K/k, Theorem F of [RW2] implies the equivalence of the Lifted Root Number Conjecture at l (see [GRW]) and a "main conjecture" of equivariant Iwasawa theory. A summary of the basic ideas is outlined in [RW2, §1] and will not be repeated here. The object here is to remove the ramification assumption on l.

Both the Lifted Root Number Conjecture at l and the "main conjecture" of equivariant Iwasawa theory are compatible with restriction and deflation. For the Lifted Root Number Conjecture this is [GRW, p.70]; for the "main conjecture" it is proved in the appendix at the end of this paper. As a consequence, we restrict ourselves to the situation

$$K = \mathbb{Q}(\zeta_n)^+ \text{ is the maximal real subfield of } \mathbb{Q}(\zeta_n) \text{ with } n = l^{m+1}n'\,,\; m \geq 0\,,\; l \nmid n'\,;\; k = \mathbb{Q}\,.$$

The tameness assumption on l in the proof of [RW2, Theorem F] was used to get a representing homomorphism of a certain class $\Omega_{(l\infty)}$ in the Grothendieck group $K_0T(\mathbb{Z}_lG)$ of the finitely generated torsion $\mathbb{Z}_lG$-modules of finite projective dimension. This representing homomorphism is a preimage of $\Omega_{(l\infty)}$ under the canonical map $\mathrm{Hom}_{\mathfrak{G}_l}(R_l(G), F_l^\times) \twoheadrightarrow K_0T(\mathbb{Z}_lG)$ (compare [GRW, Appendix A]), where, as usual, F_l is a Galois extension of $\mathbb{Q}_l$ with Galois group $\mathfrak{G}_l$, which houses the values of all characters of G, and where $R_l(G)$ is the ring of F_l-valued characters of G.

We first recall the construction of the class $\Omega_{(l\infty)}$. Let S be a finite G-stable set of primes of K, which contains all primes dividing $l\infty$ and which is sufficiently large, and let $\tilde{G}$ denote the Galois group of $K/\mathbb{Q}$. Also, S_* is a set of G-representatives of the elements in S. Define (compare [RW2, §7])

*We acknowledge financial support provided by NSERC.

$\xi = (\frac{\xi_K^{[K:\mathbb{Q}]}}{N_{K/\mathbb{Q}}\xi_K})^{t/[K:\mathbb{Q}]}$, where ξ_K is the Ramachandra number of K and where t is a suitable multiple of $[K:\mathbb{Q}]$ [1]

S-units $a(\mathfrak{p}) \in K$, for $\mathfrak{p} \in S_*$, which are fixed by $G_\mathfrak{p}$, $\Big(a(\mathfrak{p})\Big) = \mathfrak{p}^{t/f_\mathfrak{p}}$, where $f_\mathfrak{p}$ is the residue degree of $\mathfrak{p}$ with respect to $K/\mathbb{Q}$, and $a(\mathfrak{p}) \equiv 1 \bmod \mathfrak{p}'$ for $\mathfrak{p}' \neq \mathfrak{p}$ in S

a to be the $\mathbb{Q}$-idèle having component 1 except at l where the component is u^{-1} with $u \in 1 + l\mathbb{Z}_l$ defined by $\zeta_{l^\infty}^u = \zeta_{l^\infty}^\gamma$ [2] for a lift $\gamma \in G(\mathbb{Q}(\zeta_{l^\infty})/\mathbb{Q})$ of a fixed generator $\gamma_\mathbb{Q}$ of the Galois group of the cyclotomic l-extension $\mathbb{Q}_\infty \subset \mathbb{Q}(\zeta_{l^\infty})$ of $\mathbb{Q}$

$\mathcal{L}(K_\mathfrak{p}^\times)$ to be the l-completion of the multiplicative group of the local field $K_\mathfrak{p}$.

From these data we obtain a unique $\mathbb{Z}\tilde{G}$-homomorphism $\mathbb{Z}S \to \prod_S K_\mathfrak{p}^\times$ which maps $\mathfrak{p}$ to

$$\begin{cases} a(\mathfrak{p})a & \text{if } \mathfrak{p} \in S_* \text{ is finite} \\ \xi a & \text{if } \mathfrak{p} = \mathfrak{p}_\infty \end{cases} .$$

Here, $\mathfrak{p}_\infty \in S_*$ is the archimedean prime arising from $\zeta_n \mapsto e^{2\pi i/n}$. The map induces a $\mathbb{Z}_l\tilde{G}$-injection $\varphi_{(l\infty)} : \mathbb{Z}_l S_{l\infty} \rightarrowtail \prod_{S_{l\infty}} \mathcal{L}(K_\mathfrak{p}^\times)$ with $S_{l\infty} = \{\mathfrak{p} \in S : \mathfrak{p}|l \text{ or } \mathfrak{p} \text{ is archimedean}\}$ [3].

The local fundamental classes, for $\mathfrak{p} \in S_{l\infty}$, yield the exact sequence [RW2, (7.1)]

$$\prod_{S_{l\infty}} \mathcal{L}(K_\mathfrak{p}^\times) \rightarrowtail \prod_{S_{l\infty}} \mathcal{L}(V_\mathfrak{p}) \to \prod_{S_{l\infty}} \mathbb{Z}_l G_\mathfrak{p} \twoheadrightarrow \mathbb{Z}_l S_{l\infty}$$

with finitely generated $\mathbb{Z}_l G_\mathfrak{p}$-modules $\mathcal{L}(V_\mathfrak{p})$ having finite projective dimension [4]. By [RW2, Lemma 2.5] the two ends of the sequence have the same $\mathbb{Z}_l$-rank. Therefore we can perform a lifted-Ω construction (see [RW2, §1.4]) with the pair

$$\prod_{S_{l\infty}} \mathcal{L}(K_\mathfrak{p}^\times) \rightarrowtail \prod_{S_{l\infty}} \mathcal{L}(V_\mathfrak{p}) \to \prod_{S_{l\infty}} \mathbb{Z}_l G_\mathfrak{p} \twoheadrightarrow \mathbb{Z}_l S_{l\infty} \quad \text{and} \quad \varphi_{(l\infty)} : \mathbb{Z}_l S_{l\infty} \rightarrowtail \prod_{S_{l\infty}} \mathcal{L}(K_\mathfrak{p}^\times)$$

and arrive at the element $\Omega_{(l\infty)} \in K_0T(\mathbb{Z}_l G)$.

If l is tamely ramified in K/k, then $\Omega_{(l\infty)}$ is represented by $\chi \mapsto$

$$(\top_l) \qquad \begin{aligned} &(\tfrac{[K:\mathbb{Q}]\log_l a_l}{-2t})^{(\chi,1_G)}(-4t)^{[k:\mathbb{Q}]\dim V_\chi} L_l^*(1,\chi) \prod_{S_*} \det(-\mathrm{Fr}_\mathfrak{p} \mid V_\chi^{G_\mathfrak{p}^0}) \\ &\times \prod_{S_{l*}} (\, t^{\dim V_\chi^{G_\mathfrak{p}}} \det(\mathrm{Fr}_\mathfrak{p} - 1 \mid V_\chi^{G_\mathfrak{p}^0}/V_\chi^{G_\mathfrak{p}})^{-1}\,) \prod_{1\neq\psi|\chi} \rho(\psi) \end{aligned}$$

in $\mathrm{Hom}_{\mathfrak{G}_l}(R_l(G), F_l^\times)$. This is formula (8.5) in [RW2, §8] with $a_l = u^{-1}$. In it, $\mathrm{Fr}_\mathfrak{p} \in G_\mathfrak{p}/G_\mathfrak{p}^0$ denotes the Frobenius automorphism of $\mathfrak{p}$. Moreover, the factor $\prod_{1\neq\psi|\chi}\rho(\psi)$, with the product running through the non-trivial characters ψ of $\tilde{G}$ extending the given abelian character χ of G, comes from a Ramachandra index formula and need not be explained here in detail (but see e.g. [RW2, Lemma 8.1]). Finally, $L_l^*(1,\chi)$ is the residue at 1 of the l-adic L-function $L_l(s,\chi)$.

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[1] recall that $\xi_K = N_{\mathbb{Q}(\zeta_n)/K}\xi_n$ and $\xi_n = \prod_I (1-\zeta_n^{n_I})(1-\zeta_n^{-n_I})$ with I running through the proper subsets of $\{1,\ldots,s\}$, where $n = \prod_{i=1}^s p_i^{e_i}$ and $n_I = \prod_I p_i^{e_i}$

[2] meaning $\zeta_{l^n}^u = \zeta_{l^n}^\gamma$ for all natural $n \geq 1$

[3] Lemma 7.1, its corollary, Lemma 7.2 and the first paragraph of §8 in [RW2] is the reference for this.

[4] $G_\mathfrak{p}$ is the decomposition subgroup of $\mathfrak{p}$ in G. Its inertia subgroup is denoted by $G_\mathfrak{p}^0$.

The purpose of this paper is to show, without assuming that l is tame in $K = \mathbb{Q}(\zeta_n)^+$, that the homomorphism $\chi \mapsto (\top_l)$ multiplied by the function $\chi \mapsto \det(-1 \mid V_\chi^{G_{\mathfrak{l}}^0})$ represents $\Omega_{(l\infty)}$; here, $\mathfrak{l}$ is the prime above l in S_* (note that $k = \mathbb{Q}$). In fact,

THEOREM. *Let $K = \mathbb{Q}(\zeta_n)^+$ with ζ_n a primitive n th root of unity and $n = l^{m+1}n'$, $m \geq 0$, $l \nmid n'$, and let $k = \mathbb{Q}$. Assume that n' is divisible by $l-1$ when $l \neq 3$, and by 8 when $l = 3$. Then $\Omega_{(l\infty)}$ is represented by* $\chi \mapsto$

$$\begin{aligned}&\left(\tfrac{[K:\mathbb{Q}]\log_l a_l}{-2t}\right)^{(\chi,1_G)}(-4t)^{\dim V_\chi} L_l^*(1,\chi)\prod_{S_*}\det(-\mathrm{Fr}_\mathfrak{p} \mid V_\chi^{G_\mathfrak{p}^0})\\ &\quad\times \det(-1 \mid V_\chi^{G_\mathfrak{l}^0}) \cdot t^{\dim V_\chi^{G_\mathfrak{l}}} \det(\mathrm{Fr}_\mathfrak{l} - 1 \mid V_\chi^{G_\mathfrak{l}^0}/V_\chi^{G_\mathfrak{l}})^{-1} \prod_{1\neq\psi|\chi} \rho(\psi)\,.\end{aligned}$$ [5]

From the theorem it readily follows that Theorem F in [RW2] remains valid without the tameness assumption on l; we will explain this in §5.

For the proof of the theorem it will be convenient to modify the notation:

> *from now on, ζ is a primitive root of unity of order n' and ζ_m one of order l^{m+1} satisfying $\zeta_m^l = \zeta_{m-1}$.*

Our hypothesis on n' fixes the local picture (at l)

> $L_m \stackrel{\text{def}}{=} \mathbb{Q}_l(\zeta, \zeta_m)$ *is the completion of K at $\mathfrak{l} \in S_*$ above l*
>
> $L \stackrel{\text{def}}{=} \mathbb{Q}_l(\zeta)$, *the completion of $\mathbb{Q}(\zeta)$, is unramified over $\mathbb{Q}_l$ and linearly disjoint from the totally ramified extension $k_m \stackrel{\text{def}}{=} \mathbb{Q}_l(\zeta_m)$; $L_m = Lk_m$.*

Only the first statement requires an argument. Denote the complex conjugation by c. Let $\mathfrak{L} \mid \mathfrak{l}$ be primes of $\mathbb{Q}(\zeta, \zeta_m)$ respectively K above l, and let $\tilde{G}_\mathfrak{L}$, $G_\mathfrak{l}$ be the corresponding decomposition groups. Then the natural surjection $\tilde{G}_\mathfrak{L} \twoheadrightarrow G_\mathfrak{l}$ is an isomorphism if, and only if, $c \notin \tilde{G}_\mathfrak{L}$. Because of the canonical identification of decomposition groups and the Galois groups of the respective completions, the condition $c \notin \tilde{G}_\mathfrak{L}$ is equivalent to $L_m = \mathbb{Q}_l(\zeta, \zeta_m) = K_\mathfrak{l}$. Since n' is divisible by $l-1$, each primitive root ζ' of unity of order $l-1$ lies in $\mathbb{Q}(\zeta)$; also $\zeta' \in \mathbb{Q}_l$. Thus, $c \in \tilde{G}_\mathfrak{L}$ would imply that complex conjugation on $\mathbb{Q}(\zeta')$ would belong to the trivial decomposition group of the restriction of $\mathfrak{L}$ to $\mathbb{Q}(\zeta')$, which is a contradiction except when $l = 3$. However, in this case 8 divides n', hence a primitive 8 th root of unity ζ'' is then contained in $\mathbb{Q}(\zeta)$, and $c \in \tilde{G}_\mathfrak{L}$ restricted to $\mathbb{Q}(\zeta'')$ would lie in the decomposition group of the prime of $\mathbb{Q}(\zeta'')$ below $\mathfrak{L}$, which is not so.

We finally outline the basic steps in the proof of the theorem. Since $\Omega_{(l\infty)}$ is essentially a local object, we first concentrate on the contribution that comes from the local situation at l. To that end we exploit, in section 1, Coleman's power series for getting special elements of $L_m^\times$ which, in section 2, are used to define a map $f_m : \mathfrak{R}_m \rightarrowtail L_m^\times$ such that the push-out along f_m of the natural relation module sequence for the Galois group $G_l \stackrel{\text{def}}{=} G(L_m/\mathbb{Q}_l)$ [6],

$$\mathfrak{R}_m \rightarrowtail \mathbb{Z}G_l \oplus \mathbb{Z}G_l \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}\,,$$

[5]Since $\tilde{G} = G$, $\prod_{1\neq\psi|\chi}\rho(\psi)$ equals $\rho(\chi)$ or 1 according as $\chi \neq 1$ or $\chi = 1$.

[6]so $G_l = G_\mathfrak{l} = \tilde{G}_\mathfrak{L}$

tensored over $\mathbb{Z}$ with $\mathbb{Z}_l$, yields a sequence $\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V_m) \to \mathbb{Z}_l G_l \twoheadrightarrow \mathbb{Z}_l$ representing the local fundamental class. These special elements are constructed by imitating generators of $\mathfrak{R}_m$. The computation of $\operatorname{coker} f_m$ closes this section. It is perhaps worth mentioning that this is done simultaneously for all m. The next section, §3, concerns the transition to the map $\varphi_{(l\infty)}$ and §4 then the actual proof of the theorem.

At some stages of our proof, particularly in §2, similarities to work of Greither [Gr] can be observed. Greither proves Chinburg's second conjecture whereas we are concerned with a lifted version of Chinburg's third conjecture and need to verify the specific formulas of [RW2, §8] without the tameness assumption made there. For this, it would neither shorten nor ease our approach if we combined it with [Gr]. Also, the paper [BB] by Bley and Burns overlaps to some extent with ours. However, our results have been achieved independently from that paper and concentrate on the special situation introduced in [RW2], which is different from the setup in [BB].

1. Coleman power series

This section is based on [Co] with the simplification that the Lubin-Tate formal group $\mathfrak{F}$ is the multiplicative group. We recall the definitions of the trace operator $\mathfrak{S}$, the norm operator $\mathfrak{N}$, and the operator Ξ in this case [7]. Set $\mathfrak{o}$ = the ring of integers in L and σ = the Frobenius generator of $G(L/\mathbb{Q}_l)$, and let η run through the l th roots of unity.

$$\mathfrak{S} : \mathfrak{o}[[T]] \to \mathfrak{o}[[T]]\,,\ \mathfrak{S}(g)(\,(1+T)^l - 1\,) = \textstyle\sum_\eta g(\,\eta(1+T) - 1\,)$$
(compare [Co, Theorem 11])

$$\mathfrak{N} : 1 + T^2\mathfrak{o}[[T]] \to 1 + T^2\mathfrak{o}[[T]]\,,\ \mathfrak{N}(g)(\,(1+T)^l - 1\,) = \textstyle\prod_\eta g(\,\eta(1+T) - 1\,)$$
(compare [Co, Theorem 4])

$$\Xi : T^2\mathfrak{o}[[T]] \to T^2\mathfrak{o}[[T]]\,,\ \Xi(g)(T) = \sum_{i=0}^{\infty} \tfrac{1}{l^i}(\sigma^i g)(\,(1+T)^{l^i} - 1\,)$$
(compare [Co, p.110]) .

The two operators $\mathfrak{S}$ and Ξ are related by formula

$$\mathfrak{S}\Xi = \mathfrak{S} + \sigma\Xi \quad \text{on} \quad T^2\mathfrak{o}[[T]]\,. \tag{1.1a}$$

To see this, let $g \in T^2\mathfrak{o}[[T]]$:

$$\begin{aligned}
(\mathfrak{S}\Xi g)(\,(1+T)^l - 1\,) &= \textstyle\sum_\eta (\Xi g)(\,\eta(1+T) - 1\,) \\
&= \textstyle\sum_\eta \sum_{i=0}^\infty \frac{1}{l^i}(\sigma^i g)(\,(1+\eta(1+T)-1)^{l^i} - 1\,) \\
&= \textstyle\sum_\eta \sum_{i=0}^\infty \frac{1}{l^i}(\sigma^i g)(\,\eta^{l^i}(1+T)^{l^i} - 1\,) \\
&= \textstyle\sum_\eta g(\eta(1+T)-1) + l\sum_{i=0}^\infty \frac{1}{l^{i+1}}(\sigma^{i+1}g)((1+T)^{l^{i+1}} - 1) \\
&= (\mathfrak{S}g)((1+T)^l - 1) + \sigma\Big(\textstyle\sum_{i=0}^\infty \frac{(\sigma^i g)}{l^i}((1+T)^{l^{i+1}} - 1)\Big)\,.
\end{aligned}$$

Here the second term is $(\Xi g)((1+T)^l - 1)$, which establishes (1.1a).

We need some more notation. Let

$\beta \in \mathfrak{o}$ generate an integral normal basis of $L/\mathbb{Q}_l$ and have trace 1 in $\mathbb{Q}_l$

$U_L^s = 1 + \mathfrak{p}_L^s$ be the group of units of level s of a field $L \supset \mathbb{Q}_l$ with prime ideal $\mathfrak{p}_L$

[7] note that the fields K, H there are $\mathbb{Q}_l, \mathbb{Q}_l(\zeta_l)$ here

$G(\mathbb{Q}_l(\zeta_{l^\infty})/\mathbb{Q}_l) = \langle \tau, \gamma\rangle$ with τ, γ generating the subgroup of order $l-1$ respectively the Sylow-l subgroup of $G(\mathbb{Q}_l(\zeta_{l^\infty})/\mathbb{Q}_l)$

κ be the character $G(\mathbb{Q}_l(\zeta_{l^\infty})/\mathbb{Q}_l) \to \mathbb{Z}_l{}^\times$ defined by $\kappa(\tau) = t \in \mathbb{Z}_l{}^\times$ and $\kappa(\gamma) = u \in 1 + l\mathbb{Z}_l$, where $\zeta_{l^\infty}^\tau = \zeta_{l^\infty}^t$ and $\zeta_{l^\infty}^\gamma = \zeta_{l^\infty}^u$.

The group G_l has three generators σ_m $(= \sigma), \tau_m$ $(= \tau), \gamma_m$ which are chosen so that σ_m generates the Galois group of the unramified extension L_m/k_m, and τ_m and γ_m generate the subgroups of order $l-1$ respectively l^m of $G(L_m/L)$. Also, the natural identification of $G(L_m/L)$ and $G(k_m/\mathbb{Q}_l)$ sends τ_m, γ_m to the respective restrictions of τ, γ on k_m. We take the liberty of just writing σ, τ rather than σ_m, τ_m, and of viewing σ also as an element in $G(L/\mathbb{Q}_l)$ and τ, γ_m also as elements in $G(k_m/\mathbb{Q}_l)$. We finally define $\hat{\tau} = \sum_{j=0}^{l-2} \tau^j$ and similarly $\hat{\sigma}$, $\hat{\gamma}_m$.

Moreover, if Σ is a subgroup of $\langle\sigma\rangle$, then $\overline{L}_m$ and $\overline{L}$ denote the fixed fields of Σ in L_m respectively L, $\overline{\sigma}$ is the Frobenius automorphism here, and $\overline{\beta}$ is the trace of β in $\overline{L}$. Whenever we construct a particular element x in L, or a power series in $L[[T]]$, then $\overline{x}$ is meant to be the correspondingly defined element in $\overline{L}$, respectively power series in $\overline{L}[[T]]$.

In this section, by means of power series, we construct elements of $L_m^\times$ which allow us to define an injective map $\mathfrak{R}_m \to L_m^\times$ with a computable cokernel. In the course of doing so it will be advantageous to split $L_m^\times$ into its nontrivial and trivial eigenspaces with respect to the τ-action. We indicate this by adding the indices 0 and 1, respectively, at appropriate stages.

The existence of the following power series is taken from [Co]. We recall that the Galois action of $G(L(\zeta_\infty)/L) \simeq G(\mathbb{Q}_l(\zeta_\infty)/\mathbb{Q}_l)$ on a power series $f(T) \in L[[T]]$ is given by $(f(T))^\alpha = f(\,(1+T)^{\kappa(\alpha)} - 1\,)$. Note that the evaluation at $T = \zeta_m - 1$ of power series $f(T) \in 1 + T\mathfrak{o}[[T]]$ yields 1-units in L.

1. $\exists\ W_{\infty,0}(T) \in 1 + T^2\mathbb{Z}_l[[T]]$ satisfying

$$\log W_{\infty,0} = \sum_{i=0}^{\infty} \frac{1}{l^i}\Big((1+T)^{l^i\kappa(\tau\gamma)} - 1 - \kappa(\tau\gamma)(\,(1+T)^{l^i} - 1\,)\Big)$$

   and $w_{m,0} = W_{\infty,0}(T)^{l-1-\hat{\tau}}\ |_{T=\zeta_m-1} \in U^2_{k_m}$ has

$$\log w_{m,0} = (l-1-\hat{\tau})(\,\tau\gamma_m - \kappa(\tau\gamma)\,)\sum_{i=0}^{m} \frac{\zeta_m^{l^i} - 1}{l^i}\,.$$

2. $\exists\ V_{\infty,0}(T) \in 1 + T^2\mathfrak{o}[[T]]$ satisfying

$$\log V_{\infty,0} = \sum_{i=0}^{\infty} \frac{1}{l^i}\sigma^i(\beta)\Big((1+T)^{l^i\kappa(\tau\gamma)} - 1 - \kappa(\tau\gamma)(\,(1+T)^{l^i} - 1\,)\Big)$$

   and $v_{m,0}^{\sigma^m} = V_{\infty,0}(T)^{(l-1-\hat{\tau})(\tau\gamma-1)}\ |_{T=\zeta_m-1} \in U^2_{L_m}$ has

$$\log v_{m,0} = (l-1-\hat{\tau})(\tau\gamma_m - 1)(\,\tau\gamma_m - \kappa(\tau\gamma)\,)\sigma^{-m}\sum_{i=0}^{m} \frac{\sigma^i(\beta)}{l^i}(\zeta_m^{l^i} - 1)\,.$$

In fact, the existence of $W_{\infty,0}$ with $\log W_{\infty,0} = \Xi((\tau\gamma - \kappa(\tau\gamma))T)$ follows from [Co, Theorem 24] because $(\,\tau\gamma - \kappa(\tau\gamma)\,)T = (1+T)^{\kappa(\tau\gamma)} - 1 - \kappa(\tau\gamma)T \in T^2\mathbb{Z}_l[[T]]$.
For $V_{\infty,0}$ start with $\beta(\,\tau\gamma - \kappa(\tau\gamma)\,)T$.

Lemma 1.1. *The elements* $w_{m,0}$, $v_{m,0}$ *are compatible with the norms in the towers* k_m, L_m, $m \geq 0$, *respectively. Moreover,* $v_{m,0}^{\hat{\sigma}} = w_{m,0}^{\tau\gamma_m - 1}$ *and* $v_{m,0}^{\hat{\Sigma}} = \overline{v}_{m,0}$.

For the first assertion we use [Co, Theorems 4, 11 and Corollary 12] and arrive at $\mathfrak{N}(W_{\infty,0}^{l-1-\hat{\tau}}) = \mathfrak{N}(W_{\infty,0})^{l-1-\hat{\tau}}$ and $\log(\mathfrak{N}W_{\infty,0}) = \mathfrak{S}(\log W_{\infty,0})$. Therefore, in order to prove that

$$\mathfrak{N}(W_{\infty,0}^{l-1-\hat{\tau}}) \stackrel{!}{=} W_{\infty,0}^{l-1-\hat{\tau}}$$

(which implies the claim for $w_{m,0}$ by [Co, Theorem 16]), we need to check whether

1. $(l-1-\hat{\tau})\Big(\mathfrak{S}(\log W_{\infty,0}) - \log W_{\infty,0}\Big)=0$
2. $\mathfrak{N}(W_{\infty,0})$ and $W_{\infty,0}$ both have constant term 1.

Indeed, by 1., $\mathfrak{N}(W_{\infty,0}^{l-1-\hat{\tau}})$ and $W_{\infty,0}^{l-1-\hat{\tau}}$ only differ by a root of unity in L, since the logarithm of their quotient $(\mathfrak{N}W_{\infty,0}/W_{\infty,0})^{l-1-\hat{\tau}} \in 1+T\mathfrak{o}[[T]]$ vanishes. Hence, 2. guarantees the desired equality $\stackrel{!}{=}$.

For 1., we employ formula (1.1a) to compute $\mathfrak{S}\log W_{\infty,0} = \mathfrak{S}\Xi(\,(\tau\gamma - \kappa(\tau\gamma))T\,) = \mathfrak{S}(\,(\tau\gamma - \kappa(\tau\gamma))T\,) + \sigma\Xi(\,(\tau\gamma - \kappa(\tau\gamma))T\,) = \mathfrak{S}(\,(\tau\gamma - \kappa(\tau\gamma))T\,) + \log W_{\infty,0}$ since σ acts trivially. Hence it suffices to show that $l-1-\hat{\tau}$ annihilates $\mathfrak{S}(\,(\tau\gamma - \kappa(\tau\gamma)T\,)$. This follows from $\mathfrak{S}(\,(\tau\gamma - \kappa(\tau\gamma))T\,) = (\tau\gamma - \kappa(\tau\gamma))\mathfrak{S}(T)$ (by [Co, Corollary 5]) and $\mathfrak{S}(T)((1+T)^l - 1) = \sum_\eta(\eta(1+T) - 1) = -l$.

Regarding 2., we conclude from the above and from $W_{\infty,0}(0) = 1$ that $(\mathfrak{N}W_{\infty,0})^{l-1-\hat{\tau}}(0)$ is a root of unity. Now, with η as before,

$$\begin{aligned} (\mathfrak{N}W_{\infty,0})^{l-1-\hat{\tau}}(\,(1+T)^l - 1\,) &= \textstyle\prod_\eta W_{\infty,0}^{l-1-\hat{\tau}}(\,\eta(1+T) - 1\,) \\ (\mathfrak{N}W_{\infty,0})^{l-1-\hat{\tau}}(0) &= \textstyle\prod_\eta W_{\infty,0}^{l-1-\hat{\tau}}(\eta - 1) \equiv 1 \bmod (\zeta_0 - 1)^2 \quad \text{in} \quad \mathbb{Q}_l(\zeta_0)\,. \end{aligned}$$

The congruence is due to $W_{\infty,0}(T) \in 1 + T^2\mathbb{Z}_l[[T]]$. Thus the root of unity in question is =1.

We next turn to $v_{m,0}$ and follow the same reasoning as with $w_{m,0}$. However, here we need to take in the automorphism σ, since we have to show that $\mathfrak{N}V_{\infty,0}^{(l-1-\hat{\tau})(\tau\gamma-1)} = V_{\infty,0}^{(l-1-\hat{\tau})(\tau\gamma-1)\sigma}$ to apply [Co, Theorem 16]. As before

$$\begin{aligned} \mathfrak{S}(\log V_{\infty,0}) - \sigma\log V_{\infty,0} = \mathfrak{S}\Xi(\,\beta(\tau\gamma - \kappa(\tau\gamma))T\,) - \sigma\Xi(\,\beta(\tau\gamma - \kappa(\tau\gamma))T\,) = \\ \mathfrak{S}(\,\beta(\tau\gamma - \kappa(\tau\gamma))T\,) = \beta(\tau\gamma - \kappa(\tau\gamma))\mathfrak{S}(T) = \beta(\kappa(\tau\gamma) - 1)l \end{aligned}$$

is annihilated by $(l-1-\hat{\tau})(\tau\gamma - 1)$. This implies that our power series differ by a root of unity, which is 1 because $(\mathfrak{N}V_{\infty,0})(0) \equiv 1 \bmod (\zeta_0 - 1)^2$ in $L(\zeta_0)$, again as before.

The second assertion is a consequence of $V_{\infty,0}(T)^{\hat{\sigma}} = W_{\infty,0}(T)$, which holds because logarithms and leading terms agree (as $\hat{\sigma}(\beta) = 1$). Applying $(l-1-\hat{\tau})(\tau\gamma - 1)$ and evaluating at $\zeta_m - 1$ yields $v_{m,0}^{\hat{\sigma}} = w_{m,0}^{\tau\gamma_m - 1}$. Similarly $\hat{\Sigma}(\beta) = \overline{\beta}$ implies $v_{m,0}^{\hat{\Sigma}} = \overline{v}_{m,0}$ and the proof of Lemma 1.1 is complete.

In the next lemma Coleman's map Θ occurs (see [Co, p.108]). Recall also the definition of $u \in 1 + l\mathbb{Z}_l$, and set $y = \exp(\,-\frac{l-1}{l-\sigma}(\beta)\hat{\tau}(\log u)\,)$.

LEMMA 1.2. *1.* $l-\sigma$ *is a unit in* $\mathbb{Z}_l G_l$*, and* $y \in U_L^1$ *has* $y^{\hat\sigma} = u^{1-l}$, $y^{\hat\Sigma} = \overline{y}$.

2. $T^{\hat\tau(\gamma-1)} = u^{l-1} g(T)$ *with* $g(T) \in 1 + T^2\mathbb{Z}_l[[T]]$

3. $\Xi\beta\Theta \log g = \log h$ *with* $h(T) \in 1 + T^2\mathfrak{o}[[T]]$

4. $V_{\infty,1}(T) \stackrel{\text{def}}{=} y^{-1}h(T)$ *evaluated at* $T = \zeta_m - 1$ *produces the 2-unit* $v_{m,1}^{\sigma^m} \in L_m$ *satisfying*

a) $v_{m,1}^{\hat\sigma} = (\zeta_m - 1)^{\hat\tau(\gamma_m - 1)}$, $v_{m,1}^{\hat\Sigma} = \overline{v}_{m,1}$

b) $v_{m,1}^{\widehat{\tau\gamma_m}} = y^{1-\sigma}$

5. The elements $v_{m,1}$ *are compatible with the norms in the tower* L_m , $m \geq 0$.

PROOF. For 1., $l - \sigma = -(1 - l\sigma^{-1})\sigma$ has inverse $-\sigma^{-1}\sum_{i=0}^{\infty}(l\sigma^{-1})^i$. The rest is obvious.

2. follows from

$$T^{\hat\tau(\gamma-1)} = \Big(\frac{(1+T)^u - 1}{T}\Big)^{\hat\tau} = \Big(\sum_{i=1}^{\infty}\binom{u}{i}T^{i-1}\Big)^{\hat\tau} = (u(1+\tilde g(T)))^{\hat\tau} = u^{l-1}g(T)$$

with $g(T) \in 1 + T^2\mathbb{Z}_l[[T]]$ resulting from the modulo degree 2 computation

$$(1+\tilde g(T))^{\hat\tau} \equiv (1 + a_1 T)^{\hat\tau} = \prod_{j=0}^{l-2}\Big[1 + a_1\Big((1+T)^{\kappa(\tau^j)} - 1\Big)\Big] \equiv 1 + a_1\Big(\sum_{j=0}^{l-2}\kappa(\tau^j)\Big)T = 1\,,$$

since $\kappa \neq 1$ on $\langle\tau\rangle$.

By Lemma 21 (ii) of [Co] (with $h(T) = (1+T)^l - 1$), $\Theta \log g \in \mathbb{Z}_l[[T]]$. Moreover, $\Theta \log g$ is divisible by T^2, since $g - 1$ is so. Hence we can apply Ξ to $\beta\Theta\log g$ and obtain $\log h$ with some $h(T) \in 1 + T^2\mathfrak{o}[[T]]$, by [Co, Theorem 24]. This is 3.

Claim: $h^{\hat\sigma} = g$.

The claim together with 2. implies the first assertion of 4a), because $y^{\hat\sigma} = u^{1-l}$ by 1.

The claim itself is once again a consequence of Coleman's results. Indeed, $\hat\sigma \log h = \hat\sigma\Xi\beta\Theta\log g$ $= \Xi\Theta\log g$ and so $\Theta\hat\sigma\log h = \Theta\Xi\Theta\log g = \Theta\log g$ (see [Co, p.110]). Now, in the notation of [Co, p.108] this becomes, firstly, $\Theta_{\mathfrak{F}}(h^{\hat\sigma} - 1) = \Theta_{\mathfrak{F}}(g-1)$ and, secondly, by [Co, p.109/110], $\log \frac{h^{\hat\sigma}}{g} = a\log(1+T)$ for some $a \in \mathbb{Q}_l$. But the left hand side is $\equiv 0$ mod degree 2 and the right hand side is $\equiv aT$, so $a = 0$ and $\frac{h^{\hat\sigma}}{g}$ is a root of unity, whence =1, as $h, g \in 1 + T^2\mathfrak{o}[[T]]$.

The second assertion of 4a) follows analogously from $h^{\hat\Sigma} = \overline{h}$.

For 4b), observe that there is no nontrivial root of unity in U_L^1, so it suffices to verify that $\widehat{\tau\gamma_m}\log v_{m,1} = (1-\sigma)y$. For this we start from $\sigma^m \log v_{m,1} = \log V_{\infty,1}(\zeta_m - 1) = \frac{l-1}{l-\sigma}(\beta)\hat\tau(\log u) + (\log h)(\zeta_m - 1)$. Since

$$\log h(T) = \Xi\beta\Theta\log(g) = \sum_{i=0}^{\infty}\frac{\sigma^i(\beta)}{l^i}(\Theta\log g)(\,(1+T)^{l^i} - 1\,) = \sum_{i=0}^{\infty}\frac{\sigma^i(\beta)}{l^i}\Big((\log g)(\,(1+T)^{l^i} - 1\,) - \frac{1}{l}(\log g)(\,(1+T)^{l^{i+1}} - 1\,)\Big) \qquad (1.2a)$$

has most terms vanishing at $T = \zeta_m - 1$ (as $g(0) = 1$), we have

$$(\log h)(\zeta_m - 1) = \sum_{i=0}^{m}\frac{\sigma^i(\beta)}{l^i}(\log g)(\zeta_m^{l^i} - 1) - \sigma^{-1}\sum_{i=0}^{m-1}\frac{\sigma^{i+1}(\beta)}{l^{i+1}}(\log g)(\zeta_m^{l^{i+1}} - 1) =$$
$$\beta(\log g)(\zeta_m - 1) + \sum_{i=1}^{m}\frac{(1-\sigma^{-1})\sigma^i(\beta)}{l^i}(\log g)(\zeta_m^{l^i} - 1)\,.$$

But from $g(T) = (u^{-1}T^{\gamma-1})^{\hat\tau}$ this becomes

$$(\log h)(\zeta_m-1)=\beta\hat\tau(\log u^{-1}+(\gamma_m-1)\log(\zeta_m-1))+\hat\tau\sum_{i=1}^{m}\tfrac{(\sigma-1)\sigma^{i-1}(\beta)}{l^i}(\,\log u^{-1}+(\gamma_m-1)\log(\zeta_{m-i}-1)\,)\,.$$

Combining with the first equation gives

$$\sigma^m\log v_{m,1}=\hat\tau(\log u)\Big(\tfrac{\sigma-1}{l-\sigma}(\beta)-\sum_{i=1}^{m}\tfrac{(\sigma-1)\sigma^{i-1}(\beta)}{l^i}\Big)+\hat\tau(\gamma_m-1)\Big(\beta\log(\zeta_m-1)+\sum_{i=1}^{m}\tfrac{(\sigma-1)\sigma^{i-1}(\beta)}{l^i}\log(\zeta_{m-i}-1)\Big)$$

because $\frac{l-1}{l-\sigma}-1=\frac{\sigma-1}{l-\sigma}$. Acting by $\widehat{\tau\gamma_m}$ this is

$$\sigma^m\widehat{\tau\gamma_m}(\log v_{m,1}) = l^m(l-1)\hat\tau(\log u)(\sigma-1)\Big(\frac{1}{l-\sigma}-\frac{1}{l}\sum_{i=1}^{m}\Big(\frac{\sigma}{l}\Big)^{i-1}\Big)(\beta)$$

since the first term is fixed by τ and γ_m while $\widehat{\tau\gamma_m}(\gamma_m-1)=0$. Summing the geometric series we then get $\widehat{\tau\gamma_m}(\log v_{m,1}) = \sigma^{-m}l^m(l-1)\hat\tau(\log u)\Big(\frac{\sigma-1}{l-\sigma}\frac{\sigma^m}{l^m}\Big)(\beta) = (l-1)\hat\tau(\log u)(\frac{\sigma-1}{l-\sigma})(\beta) = (\sigma-1)\hat\tau(\log u)\frac{l-1}{l-\sigma}(\beta) = -(\sigma-1)\log y$, as required.

We are left with verifying 5. In analogy with the corresponding part in the proof of Lemma 1.1, it suffices to show $\mathfrak{N}V_{\infty,1}\overset{!}{=}V_{\infty,1}^{\sigma}$. We first need

$$\mathfrak{S}\log V_{\infty,1}\overset{!}{=}\sigma\log V_{\infty,1}=\sigma\Big(\frac{l-1}{l-\sigma}(\beta)\hat\tau\log u+\log h\Big)\,.$$

The left hand side equals $\mathfrak{S}(\frac{l-1}{l-\sigma}(\beta)\hat\tau\log u+\log h) = l\frac{l-1}{l-\sigma}(\beta)\hat\tau\log u+\mathfrak{S}\log h$, since $\mathfrak{S}$ transforms a constant function c into lc. Therefore, we need to show

$$(5') \qquad \mathfrak{S}\log h-\sigma\log h\overset{!}{=}\tfrac{(l-1)(\sigma-l)}{l-\sigma}(\beta)\hat\tau\log u=(l-1)\beta\hat\tau\log u^{-1}\,.$$

The left hand side in $(5')$ is, by (1.1a),

$$\mathfrak{S}\Xi\log g-\sigma\Xi\beta\Theta\log g=\mathfrak{S}\beta\Theta\log g=\beta\mathfrak{S}\Theta\log g\,.$$

However, $\Theta\log g = \Theta\hat\tau\log(u^{-1}T^{\gamma-1}) = \hat\tau\Theta(\log u^{-1}+\log T^{\gamma-1}) = \hat\tau\Big[\log u^{-1}-\frac{1}{l}\log u^{-1} + \log T^{\gamma-1}-\frac{1}{l}\log(\,(1+T)^l-1\,)^{\gamma-1}\Big]$, and applying $\mathfrak{S}$ gives, with η running through the l^{th} roots of unity,

$$\begin{aligned}&\hat\tau(l-1)\log u^{-1}+\hat\tau\textstyle\sum_\eta\Big(\log(\eta(1+T)-1)^{\gamma-1}-\frac{1}{l}\log(\,(1+\eta(1+T)-1)^l-1\,)^{\gamma-1}\Big)\\ &=\hat\tau(l-1)\log u^{-1}+\hat\tau\Big[\log\Big(\textstyle\prod_\eta(\eta(1+T)-1)^{\gamma-1}\Big)-\frac{1}{l}\log\Big(\prod_\eta((1+T)^l-1)^{\gamma-1}\Big)\Big]\\ &=\hat\tau(l-1)\log u^{-1}+\hat\tau\Big[\log((1+T)^l-1)^{\gamma-1}-\log((1+T)^l-1)^{\gamma-1}\Big]=\hat\tau(l-1)\log u^{-1}\,.\end{aligned}$$

This establishes $(5')$.

It remains to check whether $\frac{\mathfrak{N}V_{\infty,1}}{\sigma V_{\infty,1}}(0)=1$. But $\mathfrak{N}V_{\infty,1}(0)=\prod_\eta V_{\infty,1}(\eta-1)$, hence $\frac{\mathfrak{N}V_{\infty,1}}{\sigma V_{\infty,1}}(0)\equiv y^{l-\sigma} \bmod (\zeta_0-1)^2$, because $V_{\infty,1}(T)=y^{-1}h(T)$ with $h(T)\in 1+T^2\mathfrak{o}[[T]]$. But $y^{l-\sigma}$ is a 1-unit in L, hence $\equiv 1 \bmod (\zeta_0-1)^2$. Lemma 1.2 is proved.

Remark. $v_{0,1}^{\tau}=v_{0,1}$. Since $v_{m,1}$ is the value of $V_{\infty,0}(T)=y^{-1}h(T)$ at $T=\zeta_m-1$ we need to show $h(T)^\tau=h(T)$. Clearly $g(T)^\tau=g(T)$, so τ fixes the right hand side of the equality (1.2a) above, hence it fixes $\log h(T)$. Now $h(T)\in 1+T^2\mathfrak{o}[[T]]$ implies the claim.

DEFINITION. $v_m = v_{m,0}v_{m,1} \in U^2_{L_m}$, $w_m = w_{m,0}w_{m,1} \in k_m^\times$ with $w_{m,1} = (\zeta_m - 1)^{\hat\tau}$.

Combining our relations we now get

$$y^{\tau\gamma_m - 1} = 1 = w_m^{\sigma-1}\,, \quad v_m^{\hat\sigma} = w_m^{\tau\gamma_m - 1}\,, \quad v_m^{\widehat{\tau\gamma_m}} = y^{1-\sigma}\,, \tag{1.2b}$$

the last one because $\widehat{\tau\gamma_m}(\tau\gamma_m - 1) = 0$ implies $v_{m,0}^{\widehat{\tau\gamma_m}} = 1$.

The elements w_m, v_m, y are precisely the elements which we need for a map $f_m : \mathfrak{R}_m \to L_m^\times$. However, in order to calculate its cokernel we still need one more element, $r_m \in L_m$, which we now define.

LEMMA 1.3. *If $R_\infty(T) \in 1 + T\mathfrak{o}[[T]]$ is such that*

$$\log R_\infty(T) = \beta\log(1+T) + (1-\sigma)\sum_{i=0}^{\infty}\sigma^i(\beta)\Big(\frac{(1+T)^{l^i} - 1}{l^i} - \log(1+T)\Big)\,,$$

then $r_m^{\sigma_m} = R_\infty(T)^{l-1-\hat\tau}\,|_{T=\zeta_m-1} \in U^1_{L_m}$ *satisfies*

1. $\log r_m = (l-1-\hat\tau)(1-\sigma)\sum\limits_{i=0}^{m}\sigma^{i-m}(\beta)\frac{\zeta_{m-i}-1}{l^i}$
2. *The elements r_m are compatible with the norms in the tower L_m , $m \geq 0$.*
3. *$r_m^{\hat\sigma}$ is a root of unity of order l^{m+1} , $r_m^{\hat\Sigma} = \overline{r}_m$*
4. $r_m^{(\tau\gamma_m - 1)(\tau\gamma_m - \kappa(\tau\gamma))} = v_{m,0}^{1-\sigma}$

PROOF. The power series $R_\infty(T)$ is taken from [Co, p.110]. Since $R_\infty(T)-1$ is in the maximal ideal of $\mathfrak{o}[[T]]$ we have $R_\infty(0) \equiv 1 \bmod l$. Putting $T=0$ in the equation for $\log R_\infty(T)$ we get $\log R_\infty(0) = 0$ hence $R_\infty(0) = 1$. Differentiating the equation for $\log R_\infty(T)$ and putting $T = 0$ then gives $\frac{R'_\infty(0)}{R_\infty(0)} = \beta$, hence $R'_\infty(0) = \beta$. Thus $R_\infty(T) = 1 + \beta T + \dots$.

Claim 1. of Lemma 1.3 is due to $\log\zeta_m = 0$. For 2. we need to check whether $\mathfrak{N}R_\infty^{l-1-\hat\tau} = R_\infty^{(l-1-\hat\tau)\sigma}$, which is equivalent to

$$\mathfrak{S}(l-1-\hat\tau)\log R_\infty = (l-1-\hat\tau)\sigma\log R_\infty \quad\text{and}\quad \mathfrak{N}{R_\infty}^{l-1-\hat\tau}(0) = {R_\infty}^{(l-1-\hat\tau)\sigma}(0)\,.$$

This will be confirmed now. As before, η runs through the l^{th} roots of unity. We use $\log\eta = 0$ and $\sum_\eta \eta = 0$.

$$\begin{aligned}
&(\mathfrak{S}\log R_\infty)(\,(1+T)^l - 1\,) = \textstyle\sum_\eta \log R_\infty(\eta(1+T)-1)\\
&= \textstyle\sum_\eta \Big[\beta\log(\eta(1+T)) + (1-\sigma)\sum_{i=0}^\infty \sigma^i(\beta)\Big(\frac{(\eta(1+T))^{l^i}-1}{l^i} - \log(\eta(1+T))\Big)\Big]\\
&= l\beta\log(1+T) + \textstyle\sum_\eta (1-\sigma)\Big[\sum_{i=1}^\infty \sigma^i(\beta)\Big(\frac{(1+T)^{l^i}-1}{l^i} - \log(1+T)\Big)\\
&\qquad + \beta(\,\eta(1+T) - 1 - \log(1+T)\,)\Big]\\
&= l\beta\log(1+T) + l(1-\sigma)\Big[\textstyle\sum_{i=1}^\infty \sigma^i(\beta)\Big(\frac{(1+T)^{l^i}-1}{l^i} - \log(1+T)\Big]\\
&\qquad - l(1-\sigma)(\beta) - l(1-\sigma)(\beta)\log(1+T)\\
&= l\sigma(\beta)(\,1+\log(1+T)\,) - l\beta + l\Big(\log R_\infty - (1-\sigma)(\beta)(T - \log(1+T)) - \beta\log(1+T)\Big)\\
&= l\sigma(\beta)(\,1+\log(1+T)\,) - l\beta + l\log R_\infty - l(1-\sigma)(\beta)T\\
&\qquad + l(1-\sigma)(\beta)\log(1+T) - l\beta\log(1+T)\\
&= l(\sigma-1)(\beta)(1+T) + l\log R_\infty\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sigma \log R_\infty(\,(1+T)^l-1\,) = \sigma(\beta)\log(1+T)^l + \sigma(1-\sigma)\textstyle\sum_{i=0}^\infty \sigma^i(\beta)\Big(\frac{(1+T)^{l^{i+1}}-1}{l^i} - \log(1+T)^l\Big)\\
&= l\sigma(\beta)\log(1+T) + l(1-\sigma)\textstyle\sum_{i=0}^\infty \sigma^{i+1}(\beta)\Big(\frac{(1+T)^{l^{i+1}}-1}{l^{i+1}} - \log(1+T)\Big)\\
&= l\sigma(\beta)\log(1+T) + l\Big(\log R_\infty - (1-\sigma)(\beta)(T-\log(1+T)) - \beta\log(1+T)\Big)\\
&= l\sigma(\beta)\log(1+T) + l\log R_\infty - l(1-\sigma)(\beta)T + l(1-\sigma)(\beta)\log(1+T) - l\beta\log(1+T)\\
&= l\log R_\infty + l(\sigma-1)(\beta)T\,.
\end{aligned}$$

Putting the two results together shows that $\mathfrak{S}\log R_\infty - \sigma\log R_\infty = l(\sigma-1)(\beta)$, which is annihilated by $l-1-\hat\tau$. This gives the first relation by [Co, Corollary 5].

Regarding the values at zero, we know now that $(\mathfrak{N}R_\infty)^{l-1-\hat\tau}/R_\infty^{(l-1-\hat\tau)\sigma}(0)$ is a root of unity in L. By $(\mathfrak{N}R_\infty)^{l-1-\hat\tau}(0) = \prod_\eta R_\infty(\eta-1)^{l-1-\hat\tau} \equiv R_\infty(0)^{l(l-1-\hat\tau)} \bmod (\zeta_0-1)$ and $R_\infty(0)=1$, this root of unity is $\equiv 1 \bmod (\zeta_0-1)$, hence 1 because in L.

For 3. we use that $\hat\sigma\log r_m = 0$ by 1., since $\hat\sigma(1-\sigma)=0$. Thus $r_m^{\hat\sigma}$ is a root of unity in k_m. Now

$$\text{(1.3a)}\qquad \begin{array}{rcl} r_m &\equiv& (1+\beta(\zeta_m-1))^{\sigma^{-m}(l-1-\hat\tau)} \bmod (\zeta_m-1)^2\\ r_m^{\hat\sigma} &\equiv& (\,1+(\hat\sigma\beta)(\zeta_m-1)\,)^{l-1-\hat\tau} = (1+\zeta_m-1)^{l-1-\hat\tau} = \zeta_m^{l-1}/\zeta_m^{\hat\tau} = \zeta_m^{l-1}\,, \end{array}$$

because the fixed field of τ in k_m does not contain l-power roots of unity $\neq 1$. Reading the congruence $\bmod (\zeta_m-1)$ implies $r_m^{\hat\sigma}$ is a power of ζ_m; then reading it $\bmod (\zeta_m-1)^2$ gives the first part of 3. since $\zeta_m^a \equiv 1 \bmod (\zeta_m-1)^2$ implies l divides a. The second part is done analogously.

For 4. we use the identity $R_\infty(T)^{\tau\gamma-\kappa(\tau\gamma)} = V_{\infty,0}(T)^{1-\sigma}$: both sides are in $1+T\mathfrak{o}[[T]]$ and they have the same logarithm directly from the definition. Raising this identity to the power $\sigma^{-m}(\tau\gamma-1)(l-1-\hat\tau)$ and substituting $T=\zeta_m-1$ gives the result.

2. The basic sequence $\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V_m) \twoheadrightarrow \Delta G_l$

This section starts from a relation module $\mathfrak{R}_m$ for G_l, given in terms of generators and relations. We define a map $f_m : \mathfrak{R}_m \to L_m^\times$ so that the push-out of $\mathfrak{R}_m \rightarrowtail (\mathbb{Z}G_l)^2 \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}$ along f_m gives a sequence $L_m^\times \rightarrowtail V_m' \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}$ with extension class $l-1$ times the fundamental class $u_{L_m/\mathbb{Q}_l}$ of $L_m/\mathbb{Q}_l$ in $H^2(G_l, L_m^\times) = \mathrm{Ext}^2_{\mathbb{Z}G_l}(\mathbb{Z}, L_m^\times)$. We then l-complete, getting $\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V_m') \to \mathbb{Z}_lG_l \twoheadrightarrow \mathbb{Z}_l$, and pushout along multiplication by $1/l-1$ on $\mathcal{L}(L_m^\times)$. This gives the l-completion of the fundamental class sequence. Finally, we determine the cokernel of the map $\mathbb{Z}_l \otimes \mathfrak{R}_m \to \mathcal{L}(L_m^\times)$ induced by f_m.

Lemma 2.1. *Let $\mathfrak{R}_m$ be the $\mathbb{Z}G_l$-module with generators a_m, b_m, c_m and relations*

$$(\tau\gamma_m-1)b_m = 0 = (\sigma-1)c_m\,,\ \hat\sigma a_m = (\tau\gamma_m-1)c_m\,,\ \widehat{\tau\gamma_m}a_m = (1-\sigma)b_m\,.$$

Then the sequence

$$\mathfrak{R}_m \rightarrowtail \mathbb{Z}G_l \oplus \mathbb{Z}G_l \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}\,,$$

in which the first map is defined by

$$a_m \mapsto (1-\sigma, \tau\gamma_m-1)\,,\ b_m \mapsto (\widehat{\tau\gamma_m}, 0)\,,\ c_m \mapsto (0,\hat\sigma)\,,$$

the second map takes $(x,y) \in \mathbb{Z}G_l \oplus \mathbb{Z}G_l$ to $x(\tau\gamma_m-1)+y(\sigma-1) \in \mathbb{Z}G_l$ and the last map is the augmentation map, is exact.

The lemma can be checked directly or looked up in [HS, p.168]. $\mathfrak{R}_m$ is called a relation module for G_l.

LEMMA 2.2. *Let U be a subgroup of G_l and set $\overline{G} = G_l/U$. Then taking U-invariants respectively U-coinvariants transforms $\mathfrak{R}_m \rightarrowtail \mathbb{Z}G_l \oplus \mathbb{Z}G_l \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}$ into*

$$\begin{array}{ccccccc} {\mathfrak{R}_m}^U & \rightarrowtail & (\mathbb{Z}G_l \oplus \mathbb{Z}G_l)^U & \to & (\mathbb{Z}G_l)^U & & \\ & & \simeq\uparrow & & \simeq\uparrow & & \\ & & \mathbb{Z}\overline{G} \oplus \mathbb{Z}\overline{G} & \to & \mathbb{Z}\overline{G} & \twoheadrightarrow & \mathbb{Z} \end{array}$$

with the vertical arrows induced by the norm map $\hat{U}$. In particular, the resulting sequence

$${\mathfrak{R}_m}^U \rightarrowtail \mathbb{Z}\overline{G} \oplus \mathbb{Z}\overline{G} \to \mathbb{Z}\overline{G} \twoheadrightarrow \mathbb{Z}$$

shows that ${\mathfrak{R}_m}^U$ is a relation module for $\overline{G}$.

For the proof see [RW2, Lemma 3.3]. We will use the special cases:

1. $U = \Sigma \leq \langle\sigma\rangle$. We identify the corresponding $\overline{\mathfrak{R}}_m$ of Lemma 2.1 with ${\mathfrak{R}_m}^U$ by means of

$$\overline{a}_m = \hat{U}a_m\,,\ \overline{b}_m = \hat{U}b_m\,,\ \overline{c}_m = c_m\ \Big(= \hat{U}(0, \sum_{i=0}^{[G_l:U]-1} \sigma^i) \Big)\,.$$

2. $U = \langle\gamma_m^{l^i}\rangle$ with $i \leq m$. Then $\overline{G} = \langle\sigma, \tau\gamma_{m-i}\rangle$ is the Galois group of $L_{m-i}/\mathbb{Q}_l$, and $\overline{\mathfrak{R}}_m = \mathfrak{R}_{m-i}$ is identified with ${\mathfrak{R}_m}^U$ by means of

$$\overline{a}_m = \hat{U}a_m\,,\ \overline{b}_m = b_m\ \Big(= \hat{U}(\sum_{i=0}^{[G_l:U]-1} (\tau\gamma_m)^i, 0) \Big)\,,\ \overline{c}_m = \hat{U}c_m\,.$$

DEFINITION. *$f_m : \mathfrak{R}_m \to L_m^\times$ is the $\mathbb{Z}G_l$-map defined by*

$$a_m \mapsto v_m\,,\ b_m \mapsto y\,,\ c_m \mapsto w_m$$

where v_m, y, w_m are the elements introduced in the previous section.

Observe that, by (1.2b), f_m indeed respects the relations in $\mathfrak{R}_m$ and is compatible with passing to quotients $\overline{G}$ of G_l as in 1. or 2. above.

The diagram below defines the push-out of $\mathfrak{R}_m \rightarrowtail \mathbb{Z}G_l \oplus \mathbb{Z}G_l \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}$ along f_m:

$$\text{(2.2a)} \qquad \begin{array}{ccccccc} \mathfrak{R}_m & \rightarrowtail & \mathbb{Z}G_l \oplus \mathbb{Z}G_l & \to & \mathbb{Z}G_l & \twoheadrightarrow & \mathbb{Z} \\ f_m\downarrow & & \downarrow & & \| & & \| \\ L_m^\times & \rightarrowtail & V'_m & \to & \mathbb{Z}G_l & \twoheadrightarrow & \mathbb{Z} \end{array}\ .$$

Its bottom sequence has extension class a multiple, $zu_{L_m/\mathbb{Q}_l}$, of the fundamental class $u_{L_m/\mathbb{Q}_l}$ in $\langle u_{L_m/\mathbb{Q}_l}\rangle = H^2(G_l, L_m^\times) = \mathrm{Ext}^2_{\mathbb{Z}G_l}(\mathbb{Z}, L_m^\times)$. Since the fundamental class respects deflation, the analogous diagram for the level $\overline{G} = G_l/\langle\sigma\rangle$ yields

$$\begin{array}{ccccccc} \overline{\mathfrak{R}}_m & \rightarrowtail & \mathbb{Z}\overline{G} \oplus \mathbb{Z}\overline{G} & \to & \mathbb{Z}\overline{G} & \twoheadrightarrow & \mathbb{Z} \\ \overline{f}_m\downarrow & & \downarrow & & \| & & \| \\ k_m^\times & \rightarrowtail & \overline{V}'_m & \to & \mathbb{Z}\overline{G} & \twoheadrightarrow & \mathbb{Z} \end{array}$$

in which the bottom sequence has extension class $zu_{k_m/\mathbb{Q}_l}$. However, $\overline{G}$ is cyclic and $\overline{\mathfrak{R}}_m = \mathbb{Z}\overline{b}_m \oplus \mathbb{Z}\overline{G}\overline{c}_m \simeq \mathbb{Z} \oplus \mathbb{Z}\overline{G}$. This allows us to view $k_m^\times \rightarrowtail \overline{V}'_m \to \mathbb{Z}\overline{G} \twoheadrightarrow \mathbb{Z}$ as the push-out along $1 \mapsto \overline{b}_m \mapsto \overline{f}_m(\overline{b}_m) = y^{\hat{\sigma}}$ of the top sequence in

$$\begin{array}{ccccccc} \mathbb{Z} & \rightarrowtail & \mathbb{Z}\overline{G} & \to & \mathbb{Z}\overline{G} & \twoheadrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \| & & \| \\ \overline{\mathfrak{R}}_m & \rightarrowtail & \mathbb{Z}\overline{G} \oplus \mathbb{Z}\overline{G} & \to & \mathbb{Z}\overline{G} & \twoheadrightarrow & \mathbb{Z} \\ \overline{f}_m \downarrow & & \downarrow & & \| & & \| \\ k_m^\times & \rightarrowtail & \overline{V}'_m & \to & \mathbb{Z}\overline{G} & \twoheadrightarrow & \mathbb{Z} \end{array} .$$

Of course, in the top sequence $1_\mathbb{Z}$ is sent to $\widehat{\tau\gamma_m}$ and $1_{\mathbb{Z}\overline{G}}$ to $\tau\gamma_m - 1$.

By Lemma 1.3, $y^{\hat{\sigma}} = u^{1-l}$. This together with [RW1, Lemma 3.1] and [Se, p.227], see also [Sn, pp.52-53], gives $zu_{k_m/\mathbb{Q}_l} = (l-1)u_{k_m/\mathbb{Q}_l}$. Our discussion does not depend on a special choice of the level m. Hence, starting at level $m' \geq m$ produces a z' and the congruence $z' \equiv l-1 \bmod [k_{m'} : \mathbb{Q}_l] = (l-1)l^{m'}$. As a first consequence, $l-1 \mid z'$ and $\frac{z'}{l-1} \equiv 1 \bmod l^{m'}$. If now m' is such that $l^{m'}$ is the precise l-power dividing $[L_m : \mathbb{Q}_l]$, then $(\frac{z'}{l-1} - 1)u_{L_m/\mathbb{Q}_l}$ has order prime to l and thus vanishes under $H^2(G_l, L_m^\times) \to H^2(G_l, \mathcal{L}(L_m^\times))$. Consequently, the bottom sequence in the diagram below

$$\begin{array}{ccccccc} \mathfrak{R}_m & \rightarrowtail & \mathbb{Z}G_l \oplus \mathbb{Z}G_l & \to & \mathbb{Z}G_l & \twoheadrightarrow & \mathbb{Z} \\ f_m \downarrow & & & & & & \\ L_m^\times & & & & & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}(L_m^\times) & & & & & & \\ \frac{1}{l-1} \downarrow & & & & & & \\ \mathcal{L}(L_m^\times) & \rightarrowtail & \mathcal{L}(V'_m) & \to & \mathbb{Z}_l G_l & \twoheadrightarrow & \mathbb{Z}_l \end{array}$$

has the image of $u_{L_m/\mathbb{Q}_l}$ in $H^2(G_l, \mathcal{L}(L_m^\times))$ as its extension class.

Notation. The composite map $\mathbb{Z}_l \otimes_\mathbb{Z} \mathfrak{R}_m \xrightarrow{\mathbb{Z}_l \otimes f_m} \mathcal{L}(L_m^\times) \xrightarrow{1/(l-1)} \mathcal{L}(L_m^\times)$ will be denoted by $f_{m,l}$.

Lemma 2.3. *The short exact sequence $\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V'_m) \twoheadrightarrow \Delta G_l$, induced by the bottom sequence in the above diagram, coincides with the l-completion of the short exact sequence $L_m^\times \rightarrowtail V_m \twoheadrightarrow \Delta_\mathbb{Z} G_l$ having extension class $u_{L_m/\mathbb{Q}_l}$. In particular, $\mathcal{L}(V'_m)$ and $\mathcal{L}(V_m)$ can be identified and are cohomologically trivial.*

This follows directly from [RW2, 3. and 4. of Lemma 2.1], since $\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V_m) \twoheadrightarrow \Delta G_l$ and $\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V'_m) \twoheadrightarrow \Delta G_l$ have the same extension class in $\mathrm{Ext}^1_{\mathbb{Z}_l G_l}(\Delta G_l, \mathcal{L}(L_m^\times))$.

Lemma 2.4. *$f_{m,l}$ is injective and* $\operatorname{coker} f_{m,l} \simeq \mathbb{Z}_l G_l / (\,\tau\gamma_m - \kappa(\tau\gamma)\,)$

Remark. $\mathbb{Z}_l G_l / (\tau\gamma_m - \kappa(\tau\gamma)) \simeq \mathrm{ind}_{G(L_m/L)}^{G_l} \langle \zeta_m \rangle$.

Proof of Lemma: With r_m as in §1, sending 1 to $r_m^{\tau\gamma_m - 1}$ induces a homomorphism

$$\alpha_m : \mathbb{Z}_l G_l / (\tau\gamma_m - \kappa(\tau\gamma)) \to \operatorname{coker} f_{m,l}$$

because $r_m^{(\tau\gamma_m - 1)(\tau\gamma_m - \kappa(\tau\gamma))} = v_{m,0}^{1-\sigma} \in \operatorname{im} f_{m,l}$ by 4. of Lemma 1.3.

We observe that once we know α_m is surjective then the finiteness of $\operatorname{coker} f_{m,l}$ and the injectivity of $f_{m,l}$ follow immediately. The first assertion holds because $\tau\gamma_m$ having order $l^m(l-1)$ implies that $\kappa(\gamma)^{l^m(l-1)} - 1 = \kappa(\tau\gamma)^{l^m(l-1)} - (\tau\gamma_m)^{l^m(l-1)} \in \mathbb{Z}_l G_l(\tau\gamma_m - \kappa(\tau\gamma))$, hence the nonzero element $\kappa(\gamma)^{l^m(l-1)}-1$ of $\mathbb{Z}_l$ annihilates $\mathbb{Z}_l G_l/(\tau\gamma_m - \kappa(\tau\gamma))$. This implies the second assertion because $\mathbb{Z}_l \otimes \mathfrak{R}_m$ and $\mathcal{L}(L_m^\times)$ both have $\mathbb{Z}_l$-rank $|G_l|+1$.

The remainder of the proof splits into two cases. In both we first establish that α_m is surjective.

Step 1. $m = 0$

Now G_l is the group $G(L_0/\mathbb{Q}_l)$ and $\gamma_0 = 1$. The properties of elements $r_0, v_{0,0}, y, w_{0,1}$ used below are taken from §1.

Consider the $\mathbb{Z}_l G_l$-module $\mathfrak{p}_0^2$, where $\mathfrak{p}_0$ is the prime ideal in L_0. Write

$$\varepsilon_i = \frac{1}{l-1} \sum_{j \bmod (l-1)} \kappa^i(\tau^{-j})\tau^j$$

for the idempotent of $\mathbb{Z}_l\langle\tau\rangle$ associated to the character κ^i, $i \bmod (l-1)$, and put

$$a = (\tau - \kappa(\tau\gamma))(\zeta_0 - 1) \in k_0 \,.$$

Since $(\tau - \kappa(\tau\gamma))\varepsilon_i = (t^i - tu)\varepsilon_i$, we find that $\varepsilon_i(a) = t(t^{i-1}-u)\varepsilon_i(\zeta_0 - 1)$ has k_0-valuation i for $2 \le i \le l$: for the k_0-valuation of $\varepsilon_i(\zeta_0 - 1)$ can be determined from that of the Gauß sum for κ^i ([La, p.7]) and $\frac{u-1}{l} \in \mathbb{Z}_l{}^\times$. As ε_i, $2 \le i \le l$, is an $\mathfrak{o}$-basis of $\mathfrak{o}\langle\tau\rangle$ it follows that $\mathfrak{p}_0^2 = \mathfrak{o}\langle\tau\rangle \cdot a$. But $\mathfrak{o} = \mathbb{Z}_l\langle\sigma\rangle\beta$, as in §1, hence $a\beta$ is a $\mathbb{Z}_l G_l$-basis of $\mathfrak{p}_0^2$ (because σ fixes a and τ fixes β).

From $\frac{1}{l-1}\log v_{0,0} = (\tau-1)(1-\varepsilon_0)(a\beta)$, $\frac{1}{l-1}\log y = \frac{1-l}{l-\sigma}\frac{\log u}{l} l\beta$ it follows that $(1-\varepsilon_0)\mathfrak{p}_0^2 = \langle \log v_{0,0}\rangle_{\mathbb{Z}_l G_l}$, $\varepsilon_0\mathfrak{p}_0^2 = l\mathfrak{o} = \mathbb{Z}_l\langle\sigma\rangle \cdot l\beta = \langle \log y\rangle_{\mathbb{Z}_l\langle\sigma\rangle}$ because $\tau - 1$, $\frac{1-l}{l-\sigma}$ are units in $\mathbb{Z}_l\langle\tau\rangle/(\hat{\tau})$, $\mathbb{Z}_l\langle\sigma\rangle$ respectively (note that $\varepsilon_0 = \hat{\tau}/|\tau|$). Therefore $\mathfrak{p}_0^2 = (1-\varepsilon_0)\mathfrak{p}_0^2 + \varepsilon_0\mathfrak{p}_0^2 = \langle \log v_{0,0}, \log y\rangle_{\mathbb{Z}_l G_l}$. By the isomorphism $\log : U_{L_0}^2 \to \mathfrak{p}_0^2$ in the tame extension $L_0/\mathbb{Q}_l$ we thus have $U_{L_0}^2 = \langle v_{0,0}, y\rangle_{\mathbb{Z}_l G_l}$.

The map $x \mapsto \frac{x-1}{\zeta_0 - 1}$ induces an isomorphism $U_{L_0}^1/U_{L_0}^2 \to \overline{L}_0$ of $\langle\sigma\rangle$-modules, where $\overline{L}_0$ is the residue field of L_0. This respects the G_l-structure on making τ act on $\overline{L}_0$ by multiplication by $\overline{t}$, since $\kappa(\tau) = t$; then $\hat{\tau}$ acts by multiplication by $\frac{t^{l-1}-1}{t-1} = 0$, so the isomorphism takes $r_0 U_{L_0}^2$ to $-\overline{\beta}$, by (1.3a), and thus $r_0^{\tau-1}U_{L_0}^2$ to $(1-\overline{t})\overline{\beta}$. But $\overline{L}_0 = \mathbb{F}_l\langle\sigma\rangle\overline{\beta}$ and $(1-\overline{t}) \in \mathbb{F}_l\langle\sigma\rangle^\times$, so we conclude that $U_{L_0}^1 = \langle r_0^{\tau-1}, U_{L_0}^2\rangle_{\mathbb{Z}_l G_l} = \langle r_0^{\tau-1}, w_{0,1}, v_{0,0}\rangle_{\mathbb{Z}_l G_l}$.

Finally, $w_{0,1} = (\zeta_0 - 1)^{\hat{\tau}}$ has L_0-valuation $l-1$, hence

$$\mathcal{L}(L_0^\times) = \langle w_{0,1}, U_{L_0}^1\rangle_{\mathbb{Z}_l G_l} = \langle r_0^{\tau-1}, w_{0,1}, v_{0,0}, y\rangle_{\mathbb{Z}_l G_l} \,.$$

On the other hand, $\mathrm{im}\,(f_{0,l}) = \langle w_0, v_0, y\rangle_{\mathbb{Z}_l G_l}$ by definition, hence $\mathrm{im}\,(f_{0,l}) = \langle w_0, v_{0,0}, y\rangle_{\mathbb{Z}_l G_l}$ by $v_0 = v_{0,0}v_{0,1}$ with $v_{0,1} \in U_{L_0}^2 = \langle v_{0,0}, y\rangle$ and $v_0^{l-1-\hat{\tau}} = v_{0,0}^{l-1}$, which follows from the Remark following Lemma 1.2. Similarly $w_{0,0} \in U_{L_0}^2$ then implies that $\mathrm{im}\,(f_{0,l}) = \langle w_{0,1}, v_{0,0}, y\rangle$.

Combining the above we see that $r_0^{\tau-1}$ generates $\mathrm{coker}\, f_{0,l} \simeq \overline{L}_0$: for $U_{L_0}^2 \subset \mathrm{im}\,(f_{0,l})$ and $\mathcal{L}(L_0^\times)/U_{L_0}^2 = w_{0,1}^{\mathbb{Z}_l} \times U_{L_0}^1/U_{L_0}^2$. It follows that $f_{0,l}$ is injective, by the earlier argument, which also shows that $\mathbb{Z}_l G_l/(\tau - \kappa(\tau\gamma))$ is annihilated by $(\kappa(\gamma)^{l-1}-1)\mathbb{Z}_l = l\mathbb{Z}_l$, hence is isomorphic to $\mathbb{F}_l\langle\sigma\rangle$. But then α_0 is an isomorphism by $\overline{L}_0 \simeq \mathbb{F}_l\langle\sigma\rangle$.

Step 2. $m \ge 1$

Set $U = G(L_m/L_0) = \langle\gamma_m\rangle$. Deflating the l-completion of diagram (2.2a) from G_m to $G_m/U = G_0$, as in the discussion there, we observe that $(f_{m,l})^U = f_{0,l}$ up to the identification of $\mathbb{Z}_l \otimes \mathfrak{R}_m^U$ with $\mathbb{Z}_l \otimes \mathfrak{R}_0$ (and $\mathcal{L}(L_m^\times)^U$ with $\mathcal{L}(L_0^\times)$) in the second special case of Lemma 2.2. By naturality of the Tate cohomology groups H^{-1}, H^0 we get the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}_l \otimes (\mathfrak{R}_m)_U & \overset{\hat{U}}{\rightarrowtail} & \mathbb{Z}_l \otimes \mathfrak{R}_m^U & \twoheadrightarrow & H^0(U, \mathbb{Z}_l \otimes \mathfrak{R}_m) \\ (f_{m,l})_U \downarrow & & (f_{m,l})^U \downarrow & & \simeq\downarrow \\ \mathcal{L}(L_m^\times)_U & \overset{\hat{U}}{\rightarrowtail} & \mathcal{L}(L_0^\times) & \twoheadrightarrow & H^0(U, \mathcal{L}(L_m^\times)) \end{array}$$

because $H^{-1}(U, \mathbb{Z}_l \otimes \mathfrak{R}_m) = 0 = H^{-1}(U, \mathcal{L}(L_m^\times))$ as U is cyclic. Moreover, the right vertical arrow is an isomorphism because it dimension shifts in the l-completed diagram (2.2a) to the identity map on $H^{-2}(U, \mathbb{Z}_l)$ [8]. Since $(f_{m,l})^U = f_{0,l}$, Step 1 implies that $(f_{m,l})^U$ is injective, so the snake lemma gives an isomorphism

$$(\operatorname{coker} f_{m,l})_U \simeq \operatorname{coker} f_{0,l}$$

taking (the class of) $r_m^{\tau\gamma_m - 1}$ to $r_0^{\tau - 1}$ by 2. of Lemma 1.3.

Taking U-coinvariants of

$$\mathbb{Z}_l G_l/(\tau\gamma_m - \kappa(\tau\gamma)) \xrightarrow{\alpha_m} \operatorname{coker} f_{m,l} \twoheadrightarrow \operatorname{coker} \alpha_m$$

therefore shows that $(\alpha_m)_U = \alpha_0$, up to the identifications above, hence $(\operatorname{coker} \alpha_m)_U \simeq \operatorname{coker} \alpha_0 = 0$ by Step 1. Now Nakayama's lemma gives $\operatorname{coker} \alpha_m = 0$.

By our observations before Step 1, $f_{m,l}$ is thus injective. Diagram (2.2a) gives the exact sequence

$$\mathbb{Z}_l G_l \oplus \mathbb{Z}_l G_l \rightarrowtail \mathcal{L}(V'_m) \twoheadrightarrow \operatorname{coker} f_{m,l}\,,$$

so Lemma 2.3 implies that $\operatorname{coker} f_{m,l}$ is cohomologically trivial. Taking U-coinvariants of

$$\ker \alpha_m \rightarrowtail \mathbb{Z}_l G_m/(\tau\gamma_m - \kappa(\tau\gamma)) \overset{\alpha_m}{\twoheadrightarrow} \operatorname{coker} f_{m,l}$$

then gives $(\ker \alpha_m)_U = 0$, because $(\alpha_m)_U = \alpha_0$ is an isomorphism by Step 1 and $H^{-2}(U, \operatorname{coker} f_{m,l}) = 0$. Finally, Nakayama's lemma implies $\ker \alpha_m = 0$. Lemma 2.4 is established.

3. Transition to Ramachandra's φ

Recall our task, namely to find a representing homomorphism of

$$\Omega_{(l\infty)} \in K_0 T(\mathbb{Z}_l G) \quad \text{in} \quad \operatorname{Hom}_{\mathfrak{G}_l}(R_l(G), F_l^\times)\,.$$

Recall also from the introduction that $\Omega_{(l\infty)}$ arises from the lifted-Ω construction with respect to the data

$$\prod_{S_{l\infty}} \mathcal{L}(K_\mathfrak{p}^\times) \rightarrowtail \prod_{S_{l\infty}} \mathcal{L}(V_\mathfrak{p}) \to \prod_{S_{l\infty}} \mathbb{Z}_l G_\mathfrak{p} \twoheadrightarrow \mathbb{Z}_l S_{l\infty} \quad \text{and} \quad \varphi_{(l\infty)} : \mathbb{Z}_l S_{l\infty} \rightarrowtail \prod_{S_{l\infty}} \mathcal{L}(K_\mathfrak{p}^\times)\,.$$

If $\mathfrak{l} \mid l$, then, in our particular situation $K_\mathfrak{l} = L_m$ and the $\mathfrak{l}$-part of the above 4-term sequence is the sequence

$$\mathcal{L}(L_m^\times) \rightarrowtail \mathcal{L}(V_m) \to \mathbb{Z}_l G_l \oplus \mathbb{Z}_l G_l \twoheadrightarrow \mathbb{Z}_l \oplus \mathbb{Z}_l G_l$$

[8] Similarly $H^{-1}(U, \mathbb{Z}_l \otimes \mathfrak{R}_m) \simeq H^{-1}(U, \mathcal{L}(L_m^\times))$ when U is not cyclic ... .

which naturally originates in the basic sequence studied in §2. In order to get the map $\varphi_{(l\infty)}$ involved we first define a map $\iota_m : \mathbb{Z} \oplus \mathbb{Z}G_l \rightarrowtail \mathfrak{R}_m$ which, tensored with $\mathbb{Z}_l$ over $\mathbb{Z}$ and composed with $f_{m,l}$, induces

$$\operatorname{ind}(f_{m,l}\iota_{m,l}) : \operatorname{ind}_{G_l}^G(\mathbb{Z}_l \oplus \mathbb{Z}_l G_l) \rightarrowtail \oplus_x \mathcal{L}(L_m^\times) = \oplus_x \mathcal{L}(K_{x\mathfrak{l}}^\times),$$

with $x \in G$ running through a system of coset representatives of the decomposition group $G_\mathfrak{l}$ of $\mathfrak{l}$ in G. Of course, we identify $G_\mathfrak{l}$ with G_l. Observe that, since $k = \mathbb{Q}$ and K is real,

$$\operatorname{ind}_{G_l}^G(\mathbb{Z}_l \oplus \mathbb{Z}_l G_l) = \mathbb{Z}_l G/G_l \oplus \mathbb{Z}_l G \simeq \mathbb{Z}_l S_{l\infty}.$$

Now the two maps, $\operatorname{ind}(f_{m,l}\iota_{m,l})$ and $\varphi_{(l\infty)}$ are related by the triangle

$$\begin{array}{ccc} \mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G & \xleftarrow{g} & \mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G \\ \operatorname{ind}(f_{m,l}\iota_{m,l}) \searrow & & \swarrow \varphi_{(l\infty)} \\ & \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \oplus_x \mathcal{L}(K_{x\mathfrak{l}}^\times) & \end{array}$$

which also defines the map g.

By [GRW, Proposition 1] we have $\Omega_{(l\infty)} = \Omega_{\operatorname{ind}(f_{m,l}\iota_{m,l})} + \partial[\mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G_l, g]$ with ∂ the connecting homomorphism $K_1(\mathbb{Q}_l G) \xrightarrow{\partial} K_0T(\mathbb{Z}_l G)$. In Lemma 3.1 we turn to a representing homomorphism of $\Omega_{\operatorname{ind}(f_{m,l}\iota_{m,l})}$ and in Lemma 3.2 to one of $\partial[\mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G, g]$. This provides first information on $\Omega_{(l\infty)}$ which then will be exploited in the following section.

Definition. *$\iota_m : \mathbb{Z} \oplus \mathbb{Z}G_l \to \mathfrak{R}_m$ is the $\mathbb{Z}G_l$-map given by*

$$\iota_m(1,0) = \hat{\sigma} b_m\,,\ \iota_m(0,1) = a_m + \widehat{\tau\gamma_m} c_m\,.$$

On composing ι_m with the embedding $\mathfrak{R}_m \rightarrowtail \mathbb{Z}G_l \oplus \mathbb{Z}G_l$ of Lemma 2.1 it follows readily that ι_m is injective. There is a lifted-Ω construction (see [RW2, §1.4]) associated to ι_m, namely from the data

$$\mathbb{Z}_l \otimes \mathfrak{R}_m \rightarrowtail \mathbb{Z}_l G_l \oplus \mathbb{Z}_l G_l \to \mathbb{Z}_l G_l \oplus \mathbb{Z}_l G_l \twoheadrightarrow \mathbb{Z}_l \oplus \mathbb{Z}_l G_l \text{ and } \iota_{m,l} = \mathbb{Z}_l \otimes \iota_m : \mathbb{Z}_l \oplus \mathbb{Z}_l G_l \rightarrowtail \mathbb{Z}_l \otimes_\mathbb{Z} \mathfrak{R}_m\,,$$

where the first sequence is obtained from $\mathfrak{R}_m \rightarrowtail \mathbb{Z}G_l \oplus \mathbb{Z}G_l \to \mathbb{Z}G_l \twoheadrightarrow \mathbb{Z}$ (see Lemma 2.1) by adding $\mathbb{Z}G_l \xrightarrow{=} \mathbb{Z}G_l$ to its right end and then tensoring with $\mathbb{Z}_l$ over $\mathbb{Z}$.

Lemma 3.1. *1. $\Omega_{\iota_{m,l}}$ is represented*[9] *by* $\chi \mapsto \begin{cases} |G_l|^2 & \text{if } \chi = 1 \\ 1 & \text{if } \chi \neq 1 \end{cases}$ *in* $\operatorname{Hom}_{\mathfrak{G}_l}(R_l(G_l), F_l^\times)$.

2. $[\operatorname{coker} \widetilde{f_{m,l}\iota_{m,l}}] = [\operatorname{coker} \widetilde{\iota_{m,l}}] + [\operatorname{coker} f_{m,l}]$ *in* $K_0T(\mathbb{Z}_l G_l)$.

3. $\chi \mapsto \chi(\tau\gamma_m) - \kappa(\tau\gamma)$ *represents* $[\operatorname{coker} f_{m,l}]$.

The notation in 2. is defined in [RW2, §1.4]; it will also be recalled in the proof. Throughout, we drop all indices (except for the l in $\mathbb{Z}_l$ and the m in L_m, V_m) [10] and just write $\mathfrak{R}$ rather than $\mathbb{Z}_l \otimes_\mathbb{Z} \mathfrak{R}_m$. Moreover, elements in $\mathfrak{R}, \Delta G, \mathbb{Z}_l, \mathbb{Z}_l G$ are denoted by r, d, z, a, respectively, and $a(1)$ is the augmentation of a.

[9] on irreducible characters

[10] Hence G_l will be denoted by G until the end of proof. This will not cause confusion because the actual G will not appear here.

In order to get a representing homomorphism of Ω_ι we first build the pull-back diagram

$$\begin{array}{ccccc} \mathfrak{R} & \rightarrowtail & \mathfrak{R}\oplus\Delta G & \twoheadrightarrow & \Delta G \\ \| & & \tilde{\alpha}\downarrow & & |G|\downarrow \\ \mathfrak{R} & \overset{e}{\rightarrowtail} & \mathbb{Z}_lG\oplus\mathbb{Z}_lG & \twoheadrightarrow & \Delta G \end{array}$$

where $e:\mathfrak{R}\to\mathbb{Z}_lG\oplus\mathbb{Z}_lG$ is the map occurring in Lemma 2.1,

then build the push-out diagram

$$\begin{array}{ccccc} \Delta G & \rightarrowtail & \mathbb{Z}_lG\oplus\mathbb{Z}_lG & \twoheadrightarrow & \mathbb{Z}_l\oplus\mathbb{Z}_lG \\ |G|\downarrow & & \tilde{\beta}\downarrow & & \| \\ \Delta G & \rightarrowtail & (\mathbb{Z}_l\oplus\mathbb{Z}_lG)\oplus\Delta G & \twoheadrightarrow & \mathbb{Z}_l\oplus\mathbb{Z}_lG \end{array}$$

with

$$d\mapsto(d,0)\,,\ (a_1,a_2)\mapsto(a_1(1),a_2)$$

$$d\mapsto(0,d)\,,\ (z,a,d)\mapsto(z,a)$$

and define $\tilde{\iota}:\mathbb{Z}_lG\oplus\mathbb{Z}_lG\to\mathbb{Z}_lG\oplus\mathbb{Z}_lG$ by $\tilde{\iota}=\tilde{\alpha}(\iota\oplus 1)\tilde{\beta}$.
It is easily checked that above we can take

for $\tilde{\alpha}$ the map $\tilde{\alpha}(r,d)=(\hat{G}s)(d)+e(r)$ with $s:\Delta G\to\mathbb{Z}_lG\oplus\mathbb{Z}_lG$ a $\mathbb{Z}_l$-splitting of the bottom sequence in the pull-back diagram

and for $\tilde{\beta}$ the map $\tilde{\beta}(a_1,a_2)=(a_1(1),a_2,(|G|-\hat{G})a_1)$.

Thus $\tilde{\iota}$ takes

$$(1,0)\quad\text{to}\quad(\hat{G}s)(|G|-\hat{G})+\hat{\sigma}b\qquad\text{and}\qquad(0,1)\quad\text{to}\quad a+\widehat{\tau\gamma}c\,.$$

We use a calculation of [RW2, §6.3] for getting the 2×2 matrix in $(\mathbb{Z}_lG)_{2\times 2}$ of $\tilde{\iota}$:
Because $G=\langle\sigma,\tau\gamma\rangle$, we can choose $s:\Delta G\to\mathbb{Z}_lG\oplus\mathbb{Z}_lG$ to be the map

$$s\Big((\tau\gamma)^i\sigma^j-1\Big)=\Big(\frac{(\tau\gamma)^i-1}{\tau\gamma-1},(\tau\gamma)^i\frac{\sigma^j-1}{\sigma-1}\Big)\,.$$

Now, $(\hat{G}s)(|G|-\hat{G})=\sum_{g\in G}g^{-1}s(g|G|-\hat{G})$

$$\begin{aligned}&=|G|\textstyle\sum_g g^{-1}s(g-1)+\hat{G}s(|G|-\hat{G})=(|G|-\hat{G})\sum_g g^{-1}s(g-1)\\ &=(|G|-\hat{G})\textstyle\sum_{i,j}\sigma^{-j}(\tau\gamma)^{-i}\Big(\frac{(\tau\gamma)^i-1}{\tau\gamma-1},(\tau\gamma)^i\frac{\sigma^j-1}{\sigma-1}\Big)\\ &=(|G|-\hat{G})\textstyle\sum_{i,j}\Big(\sigma^{-j}(\tau\gamma)^{-i}\frac{(\tau\gamma)^i-1}{\tau\gamma-1},\sigma^{-j}\frac{\sigma^j-1}{\sigma-1}\Big)\\ &=(|G|-\hat{G})\Big(\hat{\sigma}\textstyle\sum_{i\geq 1}(\,(\tau\gamma)^{-1}+\cdots+(\tau\gamma)^{-i},|\tau\gamma|\sum_{j\geq 1}(\sigma^{-1}+\cdots+\sigma^{-j})\,)\Big)\,.\end{aligned}$$

Consequently, by Lemma 2.1 $\tilde{\iota}$ has the matrix

$$\begin{pmatrix}\hat{G}+(|G|-\hat{G})\hat{\sigma}\sum_{i\geq 1}(\,(\tau\gamma)^{-1}+\cdots+(\tau\gamma)^{-i}\,) & 1-\sigma\\ (|G|-\hat{G})|\tau\gamma|\sum_{j\geq 1}(\sigma^{-1}+\cdots+\sigma^{-j}) & \tau\gamma-1+\hat{G}\end{pmatrix}$$

Its determinant equals

$$\begin{aligned}&|G|\hat{G}+(\tau\gamma-1)(|G|-\hat{G})\hat{\sigma}\textstyle\sum_{i=1}^{|\tau\gamma|-1}(\,(\tau\gamma)^{-1}+\cdots+(\tau\gamma)^{-i}\,)-\\ &-(|G|-\hat{G})|\tau\gamma|\underbrace{(1-\sigma)\sum_{j=1}^{|\sigma|-1}(\sigma^{-1}+\cdots+\sigma^{-j})}_{=\hat{\sigma}-|\sigma|}\\ &=|G|\hat{G}+(|G|-\hat{G})\hat{\sigma}(|\tau\gamma|-\widehat{\tau\gamma})-(|G|-\hat{G})|\tau\gamma|(\hat{\sigma}-|\sigma|)\\ &=|G|\hat{G}+(|G|-\hat{G})(\hat{\sigma}|\tau\gamma|-|\tau\gamma|\hat{\sigma}+|G|)=|G|^2\,.\end{aligned}$$

Therefore, $\chi \mapsto |G|^2$ is a representing homomorphism of $[\operatorname{coker}\tilde{\imath}]$ and Ω_ι is represented as claimed, since $2\partial[\Delta G, |G|)]$ is represented by $\chi \mapsto \begin{cases} 1 & \text{if } \chi = 1 \\ |G|^2 & \text{if } \chi \neq 1 . \end{cases}$

The assertion 2. compares two lifted-Ω constructions in which the 2-extension for $\Omega_{f\iota}$ is obtained from that for Ω_ι by pushout along f as in (the l-completion of) diagram (2.2a) (with $\mathbb{Z}_l G \xrightarrow{=} \mathbb{Z}_l G$ added to its right end). This implies the commutative diagram

$$\begin{array}{rccccccc} \tilde{\imath}: & \mathbb{Z}_lG \oplus \mathbb{Z}_lG & \to & (\mathbb{Z}_l \oplus \mathbb{Z}_lG) \oplus \Delta G & \to & \mathfrak{R} \oplus \Delta G & \to & \mathbb{Z}_lG \oplus \mathbb{Z}_lG \\ & \| & & \| & & (f,1)\downarrow & & \downarrow \\ \widetilde{f\iota}: & \mathbb{Z}_lG \oplus \mathbb{Z}_lG & \to & (\mathbb{Z}_l \oplus \mathbb{Z}_lG) \oplus \Delta G & \to & \mathcal{L}(L_m^\times) \oplus \Delta G & \to & \mathcal{L}(V_m) \end{array},$$

in short

$$\begin{array}{ccccc} (\mathbb{Z}_lG)^2 & \rightarrowtail & (\mathbb{Z}_lG)^2 & \twoheadrightarrow & \operatorname{coker}\tilde{\imath} \\ \| & & \downarrow & & \downarrow \\ (\mathbb{Z}_lG)^2 & \rightarrowtail & \mathcal{L}(V_m) & \twoheadrightarrow & \operatorname{coker}\widetilde{f\iota} \\ & & \downarrow & & \\ & & \operatorname{coker} f & & \end{array}$$

which gives 2. by the snake lemma.

Finally, Lemma 2.4 implies 3. The lemma is proved.

Corollary. *$\Omega_{(l\infty)} - \partial[\mathbb{Q}_lG/G_l \oplus \mathbb{Q}_lG, g]$ is represented by*

$$\chi \mapsto \begin{cases} |G_l|^2(\chi(\tau\gamma_m) - \kappa(\tau\gamma)) & \text{if } \operatorname{res}^G_{G_l}(\chi) = 1 \\ \chi(\tau\gamma_m) - \kappa(\tau\gamma) & \text{if } \operatorname{res}^G_{G_l}(\chi) \neq 1 . \end{cases}$$

This follows from the equality $\Omega_{(l\infty)} = \Omega_{\operatorname{ind}(f_{m,l}\iota_{m,l})} + \partial[\mathbb{Q}_lG/G_l \oplus \mathbb{Q}_lG, g]$ mentioned earlier, from

$$\Omega_{f_{m,l}\iota_{m,l}} = [\operatorname{coker}\widetilde{f_{m,l}\iota_{m,l}}] - 2\partial[\Delta G_l, |G_l|] = \Omega_{\iota_{m,l}} + [\operatorname{coker} f_{m,l}]$$

and from the commutative diagram (see [Fr, p.63])

$$\begin{array}{ccc} \operatorname{Hom}_{\mathfrak{G}_l}(R_l(G), F_l^\times) & \xrightarrow{\partial} & K_0T(\mathbb{Z}_lG) \\ \operatorname{ind}^G_{G_l}\uparrow & & \operatorname{ind}^G_{G_l}\uparrow \\ \operatorname{Hom}_{\mathfrak{G}_l}(R_l(G_l), F_l^\times) & \xrightarrow{\partial} & K_0T(\mathbb{Z}_lG_l) \end{array}.$$

For the next lemma recall $v_m \in U^1_{L_m}$ from §1 and the elements $\xi, a(\mathfrak{l}) \in K$, $t \in \mathbb{N}$ as well as the $\mathbb{Q}$-idèle a from the introduction. We repeat that

> $a(\mathfrak{l}) \in \mathcal{L}(K_\mathfrak{l}^\times)$ is fixed by G_l and, when viewed in $\mathcal{L}(\mathbb{Q}_l{}^\times) \subset \mathcal{L}(K_\mathfrak{l}^\times)$, is $= l^{t/|G_l|}u^z$ for some $z \in \mathbb{Z}_l$, as u generates $1 + l\mathbb{Z}_l$
>
> a has component 1 except at l, where the component is u^{-1}.

In order to ease our notation we regard K as a subfield of $L_m = K_\mathfrak{l}$; in particular, $\log \xi$ is an unambiguous element of L_m [11]. Finally, the set $\{x\}$ is always a set of coset representatives of G_l in G, which contains $1 \in G$. Each x defines an isomorphism $K_\mathfrak{l} \to K_{x\mathfrak{l}}$ by $\mathfrak{l}\text{-}\lim_{i\to\infty} \kappa_i \mapsto x\mathfrak{l}\text{-}\lim_{i\to\infty} \kappa_i^x$ for $\kappa_i \in K$.

[11] $\log : L_m^\times \to L_m$, with $\log(l) = 0$, induces $\log : \mathcal{L}(L_m^\times) \to L_m$.

LEMMA 3.2. *For an irreducible character $\chi \in R_l(G)$ set $\chi_l = \mathrm{res}^G_{G_l}\chi$, $e_\chi = \frac{1}{|G|}\sum_{h\in G}\chi(h^{-1})h$, $e_{\chi_l} = \frac{1}{|G_l|}\sum_{h\in G_l}\chi_l(h^{-1})h$. Let g be the map defined by the commutative triangle*

$$\begin{array}{ccc} \mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G & \overset{g}{\longleftarrow} & \mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G \\ \mathrm{ind}\,(f_{m,l}\iota_{m,l}) \searrow & & \swarrow \varphi_{(l\infty)} \\ & \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \oplus_x \mathcal{L}(K^\times_{x\mathfrak{l}})\,. & \end{array}$$

Then $\partial[\mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G, g]$ is represented by

$$\chi \mapsto \begin{cases} \frac{-t|G|}{|G_l|^2} + \frac{t[\xi|\chi)}{|G_l|^2\log u} & \text{if } \chi = 1 \\ \frac{t[\xi|\chi)}{|G_l|^2\log u} & \text{if } \chi \neq 1\,,\ \chi_l = 1 \\ \frac{(l-1)[\xi|\chi)}{|G_l|(e_{\chi_l}\log v_m)} & \text{if } \chi_l \neq 1 \end{cases}$$

in $\mathrm{Hom}_{\mathfrak{G}_l}(R_l(G), F_l^\times)$, where $[\xi \mid \chi) = \sum_{g\in G}\chi(g^{-1})\log(\xi^g)$.

PROOF. We begin by evaluating g on the two generators $(\overline{1},0)$ and $(0,1)$ of $\mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G$. The respective images are denoted by (ν,c) and (μ,d) where

$$\begin{array}{l} \nu = \sum_x \nu_x\overline{x}\,,\ \mu = \sum_x \mu_x\overline{x}\ \in \mathbb{Q}_l G/G_l \quad (\text{and } x \mapsto \overline{x} \in G/G_l) \\ c = \sum_x \gamma_x x\,,\ d = \sum_x \delta_x x\ \in \mathbb{Q}_l G\,, \quad \text{so } \gamma_x, \delta_x \in \mathbb{Q}_l G_l\,. \end{array}$$

Now, the south-east arrow in the triangle takes

> $(\overline{1},0)$ to $(u^{-1},1,\dots,1)$, since $f_{m,l}(\hat\sigma b_m) = (y^{\hat\sigma})^{\frac{1}{l-1}} = u^{-1}$ (compare §2 and Lemma 1.2)
>
> $(0,1)$ to $(v_m^{\frac{1}{l-1}}l,1,\dots,1)$, since $f_{m,l}(a_m+\widehat{\tau\gamma_m}c_m) = (v_m w_m^{\widehat{\tau\gamma_m}})^{\frac{1}{l-1}}$ with $w_m = w_{m,0}w_{m,1}$, $w_{m,0}^{\hat\tau} = 1$ and $(\zeta_m-1)^{\widehat{\tau\gamma_m}} = l$.

¿From the definition of $\varphi_{(l\infty)}$, given in the introduction, we see that the south-west arrow takes

$$(\overline{1},0) \text{ to } a(\mathfrak{l})a \quad \text{and} \quad (0,1) \text{ to } \xi a\,.$$

Putting things together we obtain, in $\mathcal{L}(L_m^\times)$, the equations

$$u^{-\nu_x}v_m^{\gamma_x/l-1}l^{\gamma_x} = a(\mathfrak{l})^{x^{-1}}u^{-1}\,, \quad u^{-\mu_x}v_m^{\delta_x/l-1}l^{\delta_x} = \xi^{x^{-1}}u^{-1}\,.$$

Since (ν,c), like $(\overline{1},0)$, is fixed by G_l, all γ_x are multiples of $\hat G_l$. However, the left equation above implies that γ_x has augmentation $t/|G_l|$ or 0 according as $x=1$ or $x\neq 1$. Therefore,

$$\gamma_1 = \hat G_l\frac{t}{|G_l|^2} \quad \text{and} \quad \gamma_x = 0 \text{ for } x \neq 1\,.$$

Next we work with the right equation. In it every term except for possibly l^{δ_x} is a unit, so $l^{\delta_x} = 1$, i.e., δ_x has augmentation 0, and $v_m^{\delta_x/l-1} = \xi^{x^{-1}}u^{\mu_x-1}$ remains. We apply $\hat G_l$ and get $1 = \xi^{\hat G_l x^{-1}}u^{|G_l|(\mu_x-1)}$, whence

$$\mu_x = 1 - \frac{\hat G_l}{|G_l|}\log(\xi^{x^{-1}})/\log u\,;$$

we also apply log directly and get

$$\frac{\delta_x}{l-1}\log v_m - \log(\xi^{x^{-1}}) = (\mu_x - 1)\log u\,.$$

A representing homomorphism of $\partial[\mathbb{Q}_l G/G_l \oplus \mathbb{Q}_l G, g]$, by [GRW, p.75], is obtained from evaluating $\det\Big(g \mid \mathrm{Hom}_{F_l G}(V_\chi, F_l[G/G_l] \oplus F_l G)\Big)$, where $V_\chi = (F_l G)e_\chi$ is an $F_l G$-module affording χ. To do so we use the basis

1. $e_\chi \mapsto (\overline{e}_\chi, 0)\,,\ e_\chi \mapsto (0, e_\chi)$ if $\chi_l = 1$
2. $e_\chi \mapsto (0, e_\chi)$ if $\chi_l \neq 1\,.$

of $\mathrm{Hom}_{F_l G}(V_\chi, F_l[G/G_l] \oplus F_l G)$.

1. Case $\chi_l = 1$, so $\chi(x) = \chi(\overline{x})$:
From $g(\overline{e}_\chi, 0) = (\overline{e}_\chi \nu, e_\chi c)\,,\ g(0, e_\chi) = (\overline{e}_\chi \mu, e_\chi d)$ and from δ_x having augmentation 0 we derive the matrix

$$\begin{pmatrix} \sum_x \nu_x \chi(\overline{x}) & \sum_x \mu_x \chi(\overline{x}) \\ |G_l|\frac{t}{|G_l|^2} & 0 \end{pmatrix}$$

the determinant of which equals

$$\begin{aligned}
&-\tfrac{t}{|G_l|}\textstyle\sum_x \mu_x \chi(\overline{x}) = -\tfrac{t}{|G_l|}\sum_x \Big(1 - \tfrac{\hat{G}_l}{|G_l|}\log(\xi^{x^{-1}})/\log u\Big)\chi(\overline{x}) \\
&= -\tfrac{t}{|G_l|}\Big(\textstyle\sum_x \chi(\overline{x}) - \tfrac{1}{|G_l|\log u}\sum\limits_{x;\,h\in G_l} \chi(x)\underbrace{\chi(h)}_{=1}\log(\xi^{x^{-1}h^{-1}})\Big) \\
&= -\tfrac{t}{|G_l|}\Big(\textstyle\sum_x \chi(\overline{x}) - \tfrac{1}{|G_l|\log u}[\xi \mid \chi)\Big) = \begin{cases} -\frac{t|G|}{|G_l|^2} + \frac{t[\xi|\chi)}{|G_l|^2\log u} & \text{if } \chi = 1 \\ \frac{t[\xi|\chi)}{|G_l|^2\log u} & \text{if } \chi \neq 1\,. \end{cases}
\end{aligned}$$

2. Case $\chi_l \neq 1$:
Now $g(0, e_\chi) = (e_\chi \mu, e_\chi d) = (0, e_\chi d)$ as μ is fixed by G_l, and

$$e_\chi d = \chi(d)e_\chi = \sum_x \chi(\delta_x)\chi(x)e_\chi = \sum_x \chi_l(\delta_x)\chi(x)e_\chi\,.$$

Because $\chi_l \neq 1$, the formula relating δ_x and μ_x yields

$$\chi_l(\delta_x/l-1)(e_{\chi_l}\log v_m) - e_{\chi_l}\log(\xi^{x^{-1}}) = e_{\chi_l}(\mu_x - 1)\log u = 0\,,$$

so $\chi_l(\delta_x) = \frac{(l-1)e_{\chi_l}\log(\xi^{x^{-1}})}{e_{\chi_l}\log v_m}$ and

$$\begin{aligned}
&\textstyle\sum_x \chi_l(\delta_x)\chi(x) = \sum_x \frac{(l-1)e_{\chi_l}\log(\xi^{x^{-1}})}{e_{\chi_l}\log v_m}\chi(x) \\
&= \tfrac{l-1}{e_{\chi_l}\log v_m}\textstyle\sum_x \chi(x)|G_l|^{-1}\sum\limits_{h\in G_l}\chi(h^{-1})h\log(\xi^{x^{-1}}) \\
&= \tfrac{l-1}{|G_l|(e_{\chi_l}\log v_m)}\textstyle\sum_{x,h}\chi(xh^{-1})\log(\xi^{x^{-1}h}) = \frac{l-1}{e_{\chi_l}\log v_m}\frac{[\xi|\chi)}{|G_l|}\,.
\end{aligned}$$

The proof of Lemma 3.2 is finished. Combining the lemma with the previous corollary gives

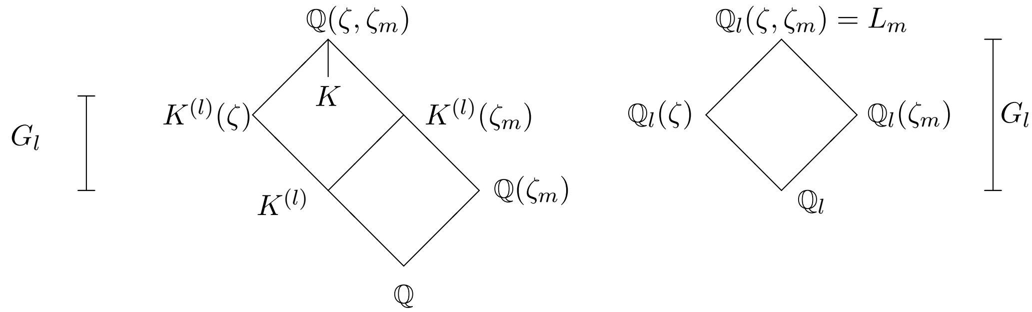
Corollary. *$\Omega_{(l\infty)}$ is represented by* $\chi \mapsto$

$$\begin{cases} (-t|G| + \frac{t[\xi|\chi)}{\log u})(1 - \kappa(\tau\gamma)) & \text{if } \chi = 1 \\ \frac{t}{\log u}(1 - \kappa(\tau\gamma))[\xi \mid \chi) & \text{if } \chi \neq 1\,,\ \chi_l = 1 \\ \frac{(l-1)[\xi|\chi)}{|G_l|(e_{\chi_l}\log v_m)}(\chi(\tau\gamma_m) - \kappa(\tau\gamma))) & \text{if } \chi_l \neq 1\,. \end{cases}$$

4. $\Omega_{(l\infty)}$

Recall that K is the fixed field of c = complex conjugation in $\mathbb{Q}(\zeta,\zeta_m)$. If $K^{(l)}$ is the fixed field of G_l in K, then $\mathbb{Q}(\zeta,\zeta_m) = K^{(l)}(\zeta)\cdot K^{(l)}(\zeta_m)$ and

$$G(\mathbb{Q}(\zeta,\zeta_m)/K^{(l)}) = G(\mathbb{Q}(\zeta,\zeta_m)/K^{(l)}(\zeta)) \times G(\mathbb{Q}(\zeta,\zeta_m)/K^{(l)}(\zeta_m))\,.$$



The characters of G, or of G_l, will always be viewed as characters of $G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q})$, respectively of $G(\mathbb{Q}(\zeta,\zeta_m)/K^{(l)})$, by means of inflation. Thus, if $\chi_l \in R_l(G_l)$ is irreducible, then χ_l can naturally be decomposed into the (tensor) product $\chi_l = \chi_\zeta \cdot \chi_m$ with χ_ζ inflated from $G(K^{(l)}(\zeta)/K^{(l)})$ and χ_m inflated from $G(K^{(l)}(\zeta_m)/K^{(l)})$.

In order to comply with the notation of [RW2, §7], it would perhaps be appropriate here to choose an embedding $s : \mathbb{Q}^c \to {\mathbb{Q}_l}^c$ of the field $\mathbb{Q}^c$ of all algebraic complex numbers into an algebraic closure ${\mathbb{Q}_l}^c$ of $\mathbb{Q}_l$ containing F_l and L_m such that s singles out the component L_m in $\mathbb{Q}_l \otimes_{\mathbb{Q}} K \subset \mathbb{Q}_l \otimes_{\mathbb{Q}} {\mathbb{Q}_l}^c$. Here, $K \subset \mathbb{Q}^c$ by means of $\zeta = e^{2\pi i/n'}$, $\zeta_m = e^{2\pi i/l^{m+1}}$. Applying s^{-1} to characters χ of G with values in F_l provides characters with values in $\mathbb{C}$ to which a Galois Gauß sum $\tau(s^{-1}\chi)$ can be associated. However, there does not seem to be a real danger of misunderstanding the notation $\tau(\chi)$ which is just short for $s\tau(s^{-1}\chi)$.

Observe that χ_m can viewed as a character of $G(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = G(\mathbb{Q}_l(\zeta_m)/\mathbb{Q}_l)$ as well.

Definition. *The conductor of χ_m is denoted by l^{j_m+1} with $j_m \geq -1$.*

In this connection we state the first lemma, the proof of which is clear.

Lemma 4.1. *Let χ_l be an irreducible F_l-character of G_l and write, as before, $\chi_l = \chi_\zeta \cdot \chi_m$. Let $e_{\chi_l}, e_{\chi_\zeta}, e_{\chi_m}$ be the corresponding primitive idempotents. Furthermore, let d_ζ, d_m be elements in $\mathbb{Q}_l(\zeta), \mathbb{Q}_l(\zeta_m)$, respectively, and set $d = d_\zeta d_m \in L_m$. Then*

$$e_{\chi_l}(d) = e_{\chi_\zeta}(d_\zeta)\cdot e_{\chi_m}(d_m) \quad in \quad {\mathbb{Q}_l}^c\,.$$

The representing homomorphism of $\Omega_{(l\infty)}$, which is displayed in the corollary at the end of the previous section, includes the terms $e_{\chi_l}\log v_m$ and $[\xi \mid \chi)$. The next lemma elaborates on $e_{\chi_l}\log v_m$. Recall that $G_l = \langle \sigma, \tau, \gamma_m \rangle$; recall also the element $\beta \in L$. The scalar product between two characters χ_1, χ_2 of a group is denoted by (χ_1, χ_2).

Lemma 4.2. *Let $\chi \in R_l(G)$ be irreducible and set $\operatorname{res}^G_{G_l}\chi = \chi_l = \chi_\zeta \cdot \chi_m$.*

1. *If $\chi_l = 1$, then $e_{\chi_l}\log v_m = 0$.*

2. If $\chi_l(\tau)\neq 1$, then $e_{\chi_l}\log v_{m,1}=0$ and

$$e_{\chi_l}\log v_{m,0}=l^{-m}(\chi(\tau\gamma_m)-1)(\,\chi(\tau\gamma_m)-\kappa(\tau\gamma)\,)\chi^{-j_m}(\sigma)(e_{\chi_\zeta}\beta)\tau(\chi_m)\ ^{12}\,.$$

3. If $\chi_l\neq 1$ but $\chi_l(\tau)=1$, then $e_{\chi_l}\log v_{m,0}=0$ and

$$e_{\chi_l}\log v_{m,1}=\begin{cases} e_{\chi_\zeta}(\beta)(\frac{1-\chi(\sigma)}{1-\chi(\sigma)/l})\frac{G_1(0)}{l^m}\,, & \chi_m=1\\ -\chi(\sigma)^{-j_m}e_{\chi_\zeta}(\beta)\frac{G_1(\chi(\gamma_m-1))}{l^m}\tau(\chi_m)\,, & \chi_m\neq 1\end{cases}$$

where $G_1(T)\in\mathbb{Z}_l[[T]]$ is the Deligne-Ribet power series defined by $L_l(1-s,1)=G_1(u^s-1)\,/\,u^s-1$, with 1 here the trivial character of $G(\mathbb{Q}(\zeta_m)/\mathbb{Q})$.

We start the proof of the lemma from $\log v_m=\log v_{m,0}+\log v_{m,1}$ and the formulas

$$\begin{aligned}\log v_{m,0}&=(l-1-\hat\tau)(\tau\gamma_m-1)(\,\tau\gamma_m-\kappa(\tau\gamma)\,)\sigma^{-m}\textstyle\sum_{i=0}^m\frac{\sigma^i(\beta)}{l^i}(\zeta_m^{l^i}-1)\\ \log v_{m,1}&=\sigma^{-m}\hat\tau\Big(\tfrac{\sigma-1}{l-\sigma}(\beta)\log u+\beta(\gamma_m-1)\log(\zeta_m-1)+\\ &\qquad+\sum_{i=1}^m\tfrac{(\sigma-1)\sigma^{i-1}(\beta)}{l^i}(\,\log u^{-1}+(\gamma_m-1)\log(\zeta_{m-i}-1)\,)\Big)\end{aligned}$$

from §1.

1. follows from $e_1\log v_m=\frac{1}{|G_l|}\log(v_m^{\hat G_l})=\frac{1}{|G_l|}\log 1=0\,.$

2. $e_{\chi_l}\hat\tau=0$ yields $e_{\chi_l}\log v_{m,1}=0$. On the other hand, $e_{\chi_m}(\zeta_{m-i}-1)=e_{\chi_m}\zeta_{m-i}$, as $\chi_m\neq 1$. Thus, on account of Lemma 4.1,

$$e_{\chi_l}\log v_{m,0}=(l-1)(\chi(\tau\gamma_m)-1)(\chi(\tau\gamma_m)-\kappa(\tau\gamma))\sum_{i=0}^m\frac{\chi(\sigma)^{-m+i}}{l^i}(e_{\chi_\zeta}\beta)(e_{\chi_m}\zeta_{m-i})\,.$$

Reading χ_m as a character of $(\mathbb{Z}/l^{m+1})^\times\simeq G(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, we see that

$$e_{\chi_m}\zeta_{m-i}=\frac{1}{(l-1)l^m}\sum_{a\in(\mathbb{Z}/l^{m+1})^\times}\chi_m(a^{-1})\zeta_{m-i}^a$$

vanishes if $m-i>j_m$, since the factor $\sum_{a\in G(\mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_{j_m}))}\zeta_{m-i}^a=\mathrm{tr}_{\,\mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_{j_m})}\zeta_{m-i}=0$ splits off. However, if $m-i<j_m$, then $e_{\chi_m}\zeta_{m-i}=0$ as well, by [La, Theorem 1.1, p.71]. So we only need to consider

$$e_{\chi_m}\zeta_{j_m}=\frac{1}{(l-1)l^m}\sum_{a\in(\mathbb{Z}/l^{m+1})^\times}\chi_m(a^{-1})\zeta_{j_m}^a=\frac{l^{m-j_m}}{(l-1)l^m}\sum_{a\in(\mathbb{Z}/l^{j_m+1})^\times}\chi_m(a^{-1})\zeta_{j_m}^a=\frac{1}{(l-1)l^{j_m}}\tau_c(\check\chi_m)\,,$$

where $\tau_c(\check\chi_m)\overset{\mathrm{def}}{=}W_{\check\chi_m}\sqrt{l^{j_m+1}}i^\delta$ is the classical Gauß sum of the contragredient character $\check\chi_m$: see [Wa, p.29]. Here, $W_{\check\chi_m}$ is the Artin root number of $\check\chi_m$, and δ is 0 or 1 according as $\chi_m(-1)=1$ or $=-1$. This classical Gauß sum is $W(\check\chi_m)\sqrt{l^{j_m+1}}i^\delta$ in the notation of [Ma, p.14]. Passing to the Galois Gauß sum $\tau(\chi_m)=W(\check\chi_m)\sqrt{l^{j_m+1}}W_\infty(\chi_m)^{-1}$, as defined in [Ma, p.48], with $W_\infty(\chi_m)=i^{-n(\chi_m,\infty)}=i^{-\delta}$ [Ma, p.24], we arrive at $e_{\chi_m}\zeta_{j_m}=\frac{1}{(l-1)l^{j_m}}\tau(\chi_m)\,,$ which is 2.

For later reference we add that, by $\tau_c(\chi_m)\tau_c(\check\chi_m)=l^{j_m+1}\chi_m(-1)$,

[12] Here $\tau(\chi_m)$ is the Galois Gauß sum of the character χ_m of $G(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ as in the Definition. Admittedly, the two occurrences of τ (as element in G_l and as Galois Gauß sum) is also a bit confusing.

$$\tau(\chi_m) = \tfrac{l^{j_m+1}}{\tau_c(\chi_m)}\chi_m(-1)\,. \tag{4.2a}$$

Finally, with respect to 3., we observe that $\chi_m(\tau) = 1$ implies $\chi_m(-1) = 1$, so χ_m is an even character. Moreover, the factor $l - 1 - \hat{\tau}$ yields $e_{\chi_l} \log v_{m,0} = 0$. On the other hand, and again by Lemma 4.1,

$$\begin{aligned} e_{\chi_l} \log v_{m,1} = (l{-}1)\chi(\sigma)^{-m}(e_{\chi_\zeta}\beta)\Big[\tfrac{\chi(\sigma-1)}{l-\chi(\sigma)}(\chi_m,1)\log u + \chi(\gamma_m{-}1)(e_{\chi_m}\log(\zeta_m{-}1)) \\ + \sum_{i=1}^{m} \tfrac{\chi(\sigma-1)\chi(\sigma)^{i-1}}{l^i}\Big((\chi_m,1)\log u^{-1} + \chi(\gamma_m - 1)(e_{\chi_m}\log(\zeta_{m-i}-1))\Big)\Big]\,. \end{aligned} \tag{4.2b}$$

When $\chi_m = 1$ the claimed formula for $e_\chi \log v_{m,1}$ follows easily from

$$\log u = \tfrac{l}{1-l}G_1(0)\,, \tag{4.2c}$$

which results from $\operatorname{res}_{s=1} L_l(s,1) = 1 - \frac{1}{l}$ and $G_1(u^s - 1) = \frac{u^{1-s}-1}{s-1} \cdot (s-1)L_l(s,1)\,.$

We may thus assume $\chi_m \neq 1$, hence $j_m \geq 0$, and compute

$$\begin{aligned} e_{\chi_m}\log(\zeta_{m-i}-1) &= \tfrac{1}{(l-1)l^m}\sum_{a\in(\mathbb{Z}/l^{m+1})^\times}\chi_m(a^{-1})\log(\zeta_{m-i}^a - 1) \\ &= \begin{cases} 0 & \text{if } m-i < j_m \\ \frac{l^i}{(l-1)l^m}\sum\limits_{a\in(\mathbb{Z}/l^{m-i+1})^\times}\chi_m(a^{-1})\log(\zeta_{m-i}^a-1) & \text{if } m-i \geq j_m\,. \end{cases} \end{aligned}$$

Above, the first equality is due to [Wa, Lemma 8.4, p.147], and the second follows from Galois theory. By [Wa, Lemma 8.6, p.148], with $f_\chi = F, g = l, t = l^{m-i-j_m}$, and [Wa, 5.18, p.63] the last expression equals

$$\frac{l^i}{(l-1)l^m}\sum_{a\in(\mathbb{Z}/l^{j_m+1})^\times}\chi_m(a^{-1})\log(\zeta_{j_m}^a - 1) = \frac{1}{(l-1)l^{m-i}}\Big(-L_l(1,\chi_m)\frac{l^{j_m+1}}{\tau_c(\chi_m)}\Big)\,.$$

Note that $\chi_m(l) = 0$ by $j_m \geq 0$. We continue and use formulas (4.2a) and 2. prior to [RW2, Proposition 5.4]. If $0 \leq i \leq m - j_m$, then

$$e_{\chi_m}\log(\zeta_{m-i}-1) = -\frac{1}{(l-1)l^{m-i}}\frac{G_{\chi_m}(0)}{\chi(\gamma_m-1)}\tau(\chi_m) = -\frac{1}{(l-1)l^{m-i}}\frac{G_1(\chi(\gamma_m)-1)}{\chi(\gamma_m-1)}\tau(\chi_m)\,.$$

Substituting this into (4.2b) we get $e_{\chi_l}\log v_{m,1} =$

$$\begin{aligned} &(l-1)\chi(\sigma)^{-m}e_{\chi_\zeta}(\beta)\chi(\gamma_m-1)\tfrac{G_1(\chi(\gamma_m)-1)}{\chi(\gamma_m-1)}\tfrac{\tau(\chi_m)}{1-l}\Big[\tfrac{1}{l^m} + \chi(\sigma-1)\sum_{i=1}^{m-j_m}\tfrac{\chi(\sigma)^{i-1}}{l^i}\tfrac{1}{l^{m-i}}\Big] \\ &= -\chi(\sigma)^{-m}e_{\chi_\zeta}(\beta)G_1(\chi(\gamma_m-1))\tfrac{\tau(\chi_m)}{l^m}\Big[1 + (\chi(\sigma)-1)\sum_{i=1}^{m-j_m}\chi(\sigma)^{i-1}\Big]\,, \end{aligned}$$

completing the proof of Lemma 4.2.

On putting the information from Lemma 4.2 together with that from the corollary at the end of the previous section we obtain that $\Omega_{(l\infty)}$ is represented by $\chi \mapsto$

$$\begin{cases} \frac{-t(l-1)}{G_1(0)l}(1-\kappa(\tau\gamma))[\xi u^{-1} \mid \chi) & \text{if } \chi_l = 1 \\ \frac{-l^m(l-1)}{|G_l|}\,\frac{(\chi(\gamma_m)-\kappa(\tau\gamma))[\xi u^{-1}|\chi)\det(\sigma^{j_m}|V_\chi/V_\chi^{G_l^0})}{e_{\chi_\zeta}(\beta)G_1(\chi(\gamma_m-1))\tau(\chi_m)\det(\frac{\sigma-1}{1-\sigma/l}|V_\chi^{G_l^0})} & \text{if } \chi_l \neq 1\,,\ \chi(\tau) = 1 \\ \frac{l^m(l-1)}{|G_l|}\,\frac{\chi(\sigma^{j_m})[\xi u^{-1}|\chi)}{e_{\chi_\zeta}(\beta)\tau(\chi_m)(\chi(\tau\gamma_m)-1)} & \text{if } \chi(\tau) \neq 1\,, \end{cases}$$

with G_l^0 the inertia subgroup of G_l. Here the first two cases of the Corollary ending §3 are unified by using $[\xi u^{-1},\chi)$, which equals $[\xi,\chi)$ unless $\chi=1$, and $\log u$ is eliminated by (4.2c). The last case splits into two cases in Lemma 4.2, with 3. there unified by the relation

$$\det(\frac{\sigma-1}{1-\sigma/l}\mid V_\chi^{G_l^0})\det(\sigma^{-j_m}\mid V_\chi/V_\chi^{G_l^0})=\begin{cases}\chi(\frac{\sigma-1}{1-\sigma/l}), & \chi_m=1\\ \chi(\sigma^{-j_m}), & \chi_m\neq 1\,.\end{cases}$$

Next we change this representing homomorphism by a factor in $\operatorname{im}(K_1(\mathbb{Z}_lG)\to K_1(\mathbb{Q}_lG))$ [13], namely by

$$\chi\mapsto\chi\Big(-G_1(\gamma_m-1)(\gamma_m-\kappa(\tau\gamma))^{-1}e_\tau+(\tau\gamma_m-1)(1-e_\tau)\Big)$$

where $e_\tau=\hat{\tau}/l-1$. Observe that $-G_1(\gamma_m-1)(\gamma_m-\kappa(\tau\gamma))^{-1}e_\tau+(\tau\gamma_m-1)(1-e_\tau)\in\mathbb{Z}_lG=\mathbb{Z}_lGe_\tau\oplus\mathbb{Z}_lG(1-e_\tau)$ is indeed a unit. This follows readily by applying the irreducible characters χ of G to it: if $\chi(\tau)\neq 1$, then $\chi(\tau\gamma_m)-1\equiv\chi(\tau)-1 \bmod (\zeta_m-1)$; if $\chi(\tau)=1$, then $\chi(\gamma_m-\kappa(\tau\gamma))\equiv 1-\kappa(\tau) \bmod (\zeta_m-1)$. Finally, $G_1(T)$ satisfies $G_1(0)=\frac{1-l}{l}\log u\in\mathbb{Z}_l^\times$, hence $G_1(T)\in\mathbb{Z}_l[[T]]^\times$ by the Weierstraß preparation theorem.

We deduce that $\Omega_{(l\infty)}$ is represented by $\chi\mapsto$

$$\begin{cases}\frac{t(l-1)}{l}[\xi u^{-1}\mid\chi) & \text{if } \chi_l=1\\ \frac{[\xi u^{-1}|\chi)\det(\sigma^{j_m}|V_\chi/V_\chi^{G_l^0})}{|\sigma|e_{\chi_\zeta}(\beta)\tau(\chi_m)\det(\frac{\sigma-1}{1-\sigma/l}|V_\chi^{G_l^0})} & \text{if } \chi_l\neq 1\,,\ \chi(\tau)=1\\ \frac{\chi(\sigma^{j_m})[\xi u^{-1}|\chi)}{|\sigma|e_{\chi_\zeta}(\beta)\tau(\chi_m)} & \text{if } \chi(\tau)\neq 1\end{cases}$$

because $|G_l|=|\sigma|l^m(l-1)$. These cases are combined to give

Proposition 4.3. *$\Omega_{(l\infty)}$ is represented by $\chi\mapsto$*

$$\frac{t^{\dim V_\chi^{G_l}}[\xi u^{-1}\mid\chi)\det(\sigma^{j_m}\mid V_\chi/V_\chi^{G_l^0})\det(1-\sigma/l\mid V_\chi^{G_l^0})}{|\sigma|e_{\chi_\zeta}(\beta)\tau(\chi_m)\det(\sigma-1\mid V_\chi^{G_l^0}/V_\chi^{G_l})}\,.$$

This is easily checked case by case, observing that $|\sigma|e_{\chi_\zeta}(\beta)=\hat{\sigma}(\beta)=1$ and $\tau(\chi_m)=1$ when $\chi_l=1$.

We next modify the formula in Proposition 4.3 to facilitate comparison with $(\top_l)$. For this purpose, we write our χ, inflated to $G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q})=G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q}(\zeta_m))\times G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q}(\zeta))$, as $\chi=\tilde{\chi}_\zeta\cdot\tilde{\chi}_m$, with $\tilde{\chi}_\zeta$ inflated from $G(\mathbb{Q}(\zeta)/\mathbb{Q})\simeq G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q}(\zeta_m))$ and $\tilde{\chi}_m$ inflated from $G(\mathbb{Q}(\zeta_m)/\mathbb{Q})\simeq G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q}(\zeta))$. We let S_ζ be the set of finite primes of $\mathbb{Q}(\zeta)$ obtained by restricting those in $\mathbb{Q}(\zeta,\zeta_m)$ above S to $\mathbb{Q}(\zeta)$. The symbol $\approx$ indicates an equality up to a factor $\det(u\mid V_\chi)$ with a unit u of $\mathbb{Z}_lG$ which is independent of χ (as at the end of §7.2 in [RW2]).

Lemma 4.4. *1.* $[\xi u^{-1}\mid\chi)\approx(\frac{|G|\log u}{2t})^{(\chi,1)}(-2t)\tau(\chi)L_l^*(1,\chi)\det(1-\frac{\sigma}{l}\mid V_\chi^{G_l^0})^{-1}\prod_{1\neq\psi|\chi}\rho(\psi)$

2. $|\sigma|e_{\chi_\zeta}(\beta)\approx\tau(\tilde{\chi}_\zeta)\prod_{\mathfrak{p}\in S_{\zeta,*}}\det(-\mathrm{Fr}_\mathfrak{p}\mid V_{(\tilde{\chi}_\zeta)_\mathfrak{p}}^{D_\mathfrak{p}^0})^{-1}$ *with $(\tilde{\chi}_\zeta)_\mathfrak{p}$ denoting the restriction of $\tilde{\chi}_\zeta$ to the decomposition group $D_\mathfrak{p}$ of $\mathfrak{p}$ in $G(\mathbb{Q}(\zeta)/\mathbb{Q})$.*

[13] note that the elements in this image represent the trivial element in $K_0T(\mathbb{Z}_lG)$

3. $\tau(\chi_m)=\tau(\tilde\chi_m)$ *and* $\tau(\chi)=\tau(\tilde\chi_\zeta)\tau(\tilde\chi_m)\chi(\sigma^{-j_m-1})\chi_m(f_{\tilde\chi_\zeta})^{-1}$.

The assertion 1. is [RW2, Corollary in §8 [14]], with $a_l=u^{-1}$, because $\tilde G=G$ and

$$\det(\sum_{x\in G} sx\log(\xi u^{-1})x^{-1}\mid V_\chi)=\det(\sum_{x\in G}\log_l(s(\xi^x)u^{-1})x\mid V_\chi)$$

is $[\xi u^{-1}\mid\chi)$ in our present notation in which s is omitted and $\log_l$ is written log (compare with the notation before Lemmas 7.8 and 8.1 of [RW2]).

For 2., we apply [RW2, Theorem E [15]] to the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ (in place of K/k there). Then $\tilde\chi_\zeta$ restricts to χ_ζ, $\prod_{\mathfrak{p}\in S_{l*}}\mathcal{N}_{s_\mathfrak{p}k_\mathfrak{p}/\mathbb{Q}_l}(s_\mathfrak{p}(b_\mathfrak{p})|\chi'_\mathfrak{p})$ becomes $\sum_{i=0}^{|\sigma|-1}\chi_\zeta(\sigma^{-i})\sigma^i(\beta)=|\sigma|e_{\chi_\zeta}(\beta)$ and S_ζ is an admissible S'.

For 3. we first note that $\tilde\chi_m$ is the inflation of χ_m [16] under $G(\mathbb{Q}(\zeta,\zeta_m)/\mathbb{Q})\to G(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. So the first equality is due to the invariance of our Galois Gauß sums with respect to inflation, as follows from [Ma, pp.18,22,48], and from the invariance of $W_\infty(\omega)=1$ or i^{-1} according as $\omega(-1)=1$ or $=-1$ (see [Ma, p.24]), which is preserved for $\omega=\chi_m$ since the complex conjugation on $\mathbb{Q}(\zeta,\zeta_m)$ has nontrivial restriction to $\mathbb{Q}(\zeta_m)$.

Actually we also need to observe that the above, with $\omega=\chi,\tilde\chi_\zeta$, also applies to the invariance of the Galois Gauß sums in 1. and 2.

Since the conductors $f_{\tilde\chi_\zeta}$ and l^{j_m+1} of $\tilde\chi_\zeta$, $\tilde\chi_m$, respectively, are coprime, we have

$$\tau(\chi)=\tau(\tilde\chi_\zeta)\tau(\tilde\chi_m)\tilde\chi_\zeta(l^{j_m+1})^{-1}\tilde\chi_m(f_{\tilde\chi_\zeta})^{-1}.$$

Observing that $\sigma\in G_l$ induces the Frobenius automorphism $\zeta\mapsto\zeta^l$ on $\mathbb{Q}(\zeta)/\mathbb{Q}$, hence $\tilde\chi_\zeta(l)=\chi(\sigma)$, and that $\tilde\chi_m$ inflates χ_m, the lemma is proved.

Substituting 1.,2. of the lemma into Proposition 4.3 and taking account of the cancellations provided by 3. and by $\det(\sigma\mid V_\chi/V_\chi^{G_l^0})=\chi(\sigma)^{1-(\chi_m,1)}$, $j_m(\chi_m,1)=-(\chi_m,1)$, we obtain the

Corollary. $\Omega_{(l\infty)}$ *is represented by* $\chi\mapsto$

$$t^{\dim V_\chi^{G_l}}\left(\frac{|G|\log u}{2t}\right)^{(\chi,1)}(-2t)L_l^*(1,\chi)\chi(\sigma)^{(\chi_m,1)-1}\chi_m(f_{\tilde\chi_\zeta})^{-1}$$
$$\times\ \det(\sigma-1\mid V_\chi^{G_l^0}/V_\chi^{G_l})^{-1}\prod_{\mathfrak{p}\in S_{\zeta,*}}\det(-\mathrm{Fr}_\mathfrak{p}\mid V_{(\tilde\chi_\zeta)_\mathfrak{p}}^{D_\mathfrak{p}^0})\prod_{1\neq\psi|\chi}\rho(\psi).$$

This is to be compared with the χ-value in formula $(\top_l)$ in the introduction, which in our special case $k=\mathbb{Q}$, $K=\mathbb{Q}(\zeta_n)^+$ reads

$$(\top'_l)\qquad t^{\dim V_\chi^{G_l}}\left(\frac{|G|\log u}{2t}\right)^{(\chi,1)}(-4t)L_l^*(1,\chi)\det(\sigma-1\mid V_\chi^{G_l^0}/V_\chi^{G_l})^{-1}$$
$$\times\prod_{\mathfrak{p}\in S_*}\det(-\mathrm{Fr}_\mathfrak{p}\mid V_\chi^{G_\mathfrak{p}^0})\prod_{1\neq\psi|\chi}\rho(\psi)$$

because $a_l=u^{-1}$, $S_{l*}=\{\mathfrak{l}\}$ and $\sigma=\mathrm{Fr}_\mathfrak{l}$ on $V_\chi^{G_\mathfrak{p}^0}$. We analyze the difference in these formulas.

[14] in which the first $\log_l$ should read log

[15] whose proof neither requires K to be real nor χ to be even

[16] in the notation of 2. of Lemma 4.2

LEMMA 4.5. *For prime numbers $p \neq l$, let F_p denote the automorphism $\zeta \mapsto \zeta\,,\ \zeta_m \mapsto {\zeta_m}^p$ of $\mathbb{Q}(\zeta, \zeta_m)$. Then*

1. $$\frac{\prod_{\mathfrak{p}\in S_{\zeta,*}} \det(-\mathrm{Fr}_{\mathfrak{p}}|V_{(\tilde{\chi}_\zeta)_{\mathfrak{p}}}^{D_{\mathfrak{p}}^0})}{\prod_{\mathfrak{p}\in S_*} \det(-\mathrm{Fr}_{\mathfrak{p}}|V_\chi^{G_{\mathfrak{p}}^0})} = \chi(-\sigma)^{1-(\chi_m,1)} \prod_{\mathfrak{l}\neq\mathfrak{p}\in S_*} \det(F_p \mid V_\chi^{G_{\mathfrak{p}}^0})^{-1}$$
2. $$\chi_m(f_{\tilde{\chi}_\zeta}) \approx \prod_{\mathfrak{l}\neq\mathfrak{p}\in S_*} \det(F_p \mid V_\chi^{G_{\mathfrak{p}}^0})^{-1}$$

Choose a prime $\mathfrak{P}$ of $\mathbb{Q}(\zeta, \zeta_m)$ above each $\mathfrak{p} \in S_*$ and set $\mathfrak{P}_\zeta = \mathfrak{P} \cap \mathbb{Q}(\zeta)$. Putting $G' = G(\mathbb{Q}(\zeta, \zeta_m)/\mathbb{Q})$ we have

$$\begin{array}{ccccc} \langle c \rangle & \rightarrowtail & G' & \twoheadrightarrow & G \\ & & \cup & & \cup \\ & & (G')_{\mathfrak{P}}^0 & \twoheadrightarrow & G_{\mathfrak{p}}^0 \end{array}$$

$$\operatorname{res}^{G}_{G_{\mathfrak{p}}^0}\chi = 1 \iff \operatorname{res}^{G'}_{(G')_{\mathfrak{P}}^0}\chi = 1$$

because χ is real (inflated to G').

For 1., note that $\mathfrak{p} \leftrightarrow \mathfrak{P}_\zeta$ is a bijection $S_* \leftrightarrow S_{\zeta,*}$, so we may study the contribution of each $\mathfrak{p}$ to our ratio $\prod_{S_{\zeta,*}} / \prod_{S_*}$.

If $\mathfrak{p} \neq \mathfrak{l}$, then $(G')_{\mathfrak{P}}^0 \leq G((\mathbb{Q}(\zeta, \zeta_m)/\mathbb{Q}(\zeta_m))$ hence $[\operatorname{res}^{G}_{G_{\mathfrak{p}}^0}\chi = 1 \iff \operatorname{res}^{G'}_{(G')_{\mathfrak{P}}^0}\tilde{\chi}_\zeta = 1]$, and for such $\mathfrak{p}$ we have

$$\det(-\mathrm{Fr}_{\mathfrak{p}} \mid V_\chi^{G_{\mathfrak{p}}^0}) = -\chi(\mathrm{Fr}_{\mathfrak{p}}) = -\tilde{\chi}_\zeta(\mathrm{Fr}_{\mathfrak{P}})\tilde{\chi}_m(\mathrm{Fr}_{\mathfrak{P}}) = \det(-\mathrm{Fr}_{\mathfrak{P}_\zeta} \mid V_{(\tilde{\chi}_\zeta)_{\mathfrak{P}_\zeta}}^{D_{\mathfrak{P}_\zeta}^0})\tilde{\chi}(F_p)$$

because $\mathrm{Fr}_{\mathfrak{P}}$ and F_p agree on $\mathbb{Q}(\zeta_m)$; thus the $\mathfrak{p}$-contribution to the ratio is $\tilde{\chi}_m(F_p)^{-1} = \chi(F_p)^{-1}$ in this case. In the other case, $\operatorname{res}^{G}_{G_{\mathfrak{p}}^0}\chi \neq 1$, it is 1, which is what $\det(F_p \mid V_\chi^{G_{\mathfrak{p}}^0})^{-1}$ gives in both cases.

When $\mathfrak{p} = \mathfrak{l}$, then $(G')_{\mathfrak{P}}^0 = G(\mathbb{Q}(\zeta, \zeta_m)/\mathbb{Q}(\zeta))$ so $[\operatorname{res}^{G}_{G_{\mathfrak{p}}^0}\chi = 1 \iff \tilde{\chi}_m = 1]$. If $\tilde{\chi}_m = 1$ then $\chi(-\mathrm{Fr}_{\mathfrak{l}}) = \tilde{\chi}_\zeta(-\mathrm{Fr}_{\mathfrak{P}_\zeta})$ hence $\mathfrak{l}$ contributes 1 to the ratio $\prod_{S_{\zeta,*}} / \prod_{S_*}$. If $\tilde{\chi}_m \neq 1$ then $\mathfrak{l}$ contributes 1 to $\prod_{S_*}$, but the contribution of $\mathfrak{P}_\zeta$ to $\prod_{S_{\zeta,*}}$ is still $\det(-\mathrm{Fr}_{\mathfrak{P}_\zeta} \mid V_{(\tilde{\chi})_{\mathfrak{P}_\zeta}}^{D_{\mathfrak{P}_\zeta}^0}) = \chi(-\sigma)$, because σ is $\mathrm{Fr}_{\mathfrak{P}_\zeta}$ on $\mathbb{Q}(\zeta)$ and the identity on $\mathbb{Q}(\zeta_m)$.

For assertion 2., factor $n' = \prod_p p^{n_p}$, noting that each $p \neq l$ and that only such $p|n'$ can contribute to $f_{\tilde{\chi}_\zeta}$. Correspondingly, each $\mathfrak{p} \in S_*$ not dividing ln' has $G_{\mathfrak{p}}^0 = 1$, hence $\det(F_p \mid V_\chi^{G_{\mathfrak{p}}^0}) = \chi(F_p) \approx 1$ because the image of F_p in $\mathbb{Z}_l G$ is a unit. So we may again focus on the p-parts of our formula for $p|n'$, and must show $\chi_m(f_{\tilde{\chi}_\zeta}^{(p)}) \approx \det(F_p \mid V_\chi^{G_{\mathfrak{p}}^0})^{-1}$.

The inertia subgroup $(G')_{\mathfrak{P}}^0$ of $\mathfrak{P}|p$ is $G(\mathbb{Q}(\zeta, \zeta_m)/\mathbb{Q}(\zeta^{p^{n_p}}, \zeta_m)) \simeq (\mathbb{Z}/p^{n_p}\mathbb{Z})^\times$ and the ν^{th} ramification group $(G')_{\mathfrak{P}}^\nu$ corresponds to $\{a \in (\mathbb{Z}/p^{n_p}\mathbb{Z})^\times : a \equiv 1 \bmod p^\nu\}$ for $1 \leq \nu \leq n_p$ ([Se, p.86]). Define

$$\begin{aligned} e'_{p,\nu} &= p^{\nu-n_p}\widehat{(G')_{\mathfrak{P}}^\nu} \in \mathbb{Z}_l G'\,,\ 1 \leq \nu \leq n_p \\ \varepsilon'_p &= F_p^{n_p} + \textstyle\sum_{\nu=1}^{n_p-1} e'_{p,\nu}(F_p^\nu - F_p^{\nu+1})\,. \end{aligned}$$

These ε'_p are units in $\mathbb{Z}_p G'$: applying an irreducible character χ' of G' gives

$$\chi'(\varepsilon'_p) = \chi'(F_p^{t_p})$$

with $t_p = t_p(\chi') \geq 1$ minimal with $\operatorname{res}^{G'}_{(G')_{\mathfrak{P}}^{t_p}}\chi' = 1$.

Letting ε_p be the image of ε'_p in $(\mathbb{Z}_pG)^\times$, it follows that $\det(\varepsilon_p \mid V_\chi^{G^0_\mathfrak{p}}) = \det(F_p \mid V_\chi^{G^0_\mathfrak{p}})$ and that

$$\det(\varepsilon_p \mid V_\chi / V_\chi^{G^0_\mathfrak{p}}) = \begin{cases} 1 & , \ \operatorname{res}^G_{G^0_\mathfrak{p}} \chi = 1 \\ \chi(F_p^{t_p}) & , \ \operatorname{res}^G_{G^0_\mathfrak{p}} \chi \neq 1 \,. \end{cases}$$

This implies $\chi_m(f^{(p)}_{\tilde\chi_\zeta}) = \det(\varepsilon_p \mid V_\chi / V_\chi^{G^0_\mathfrak{p}})$. For $[\operatorname{res}^G_{G^0_\mathfrak{p}} \chi \neq 1 \iff \operatorname{res}^{G'}_{(G')^0_\mathfrak{P}} \chi \neq 1]$, and in this case $f^{(p)}_{\tilde\chi_\zeta} = p^{t_p}$ ([Se, pp.109,81/82]), hence $\chi_m(f^{(p)}_{\tilde\chi_\zeta}) = \chi_m(\mathrm{Fr}_\mathfrak{p}{}^{t_p}) = \chi(F_p^{t_p}) = \det(\varepsilon_p \mid V_\chi / V_\chi^{G^0_\mathfrak{p}})$. In the other case, $V_\chi^{G^0_\mathfrak{p}} = V_\chi$ and $f^{(p)}_{\tilde\chi_\zeta} = 1$ since χ trivial on $(G')^0_\mathfrak{P}$ implies $\tilde\chi_\zeta$ trivial on the inertia group of $\mathfrak{P}_\zeta$ in $\mathbb{Q}(\zeta)/\mathbb{Q}$.

But $\chi \mapsto \det(\varepsilon_p \mid V_\chi)$ is ≈ 1, by $\varepsilon_p \in (\mathbb{Z}_lG)^\times$, hence $\chi(f^{(p)}_{\tilde\chi_\zeta}) \approx \det(\varepsilon_p \mid V_\chi^{G^0_\mathfrak{p}})^{-1} = \det(F_p \mid V_\chi^{G^0_\mathfrak{p}})^{-1}$ and the lemma is proved.

We finally prove the Theorem. Dividing the formula of the last corollary by that of $(\top'_l)$ and applying the lemma gives

$$\begin{aligned} &\tfrac{1}{2}\chi(\sigma)^{(\chi_m,1)-1}\chi_m(f_{\tilde\chi_\zeta})^{-1} \frac{\prod_{S_{\zeta,*}} \det(-\mathrm{Fr}_\mathfrak{p} | V_{(\tilde\chi_\zeta)_\mathfrak{p}}^{D^0_\mathfrak{p}})}{\prod_{S_*} \det(-\mathrm{Fr}_\mathfrak{p} | V_\chi^{G^0_\mathfrak{p}})} \\ \approx\ & \tfrac{1}{2}\chi(\sigma)^{(\chi_m,1)-1}\chi(-\sigma)^{1-(\chi_m,1)} = \frac{(-1)^{(\chi_m,1)}}{-2} \approx (-1)^{(\chi_m,1)} \,. \end{aligned}$$

Since $\det(-1 \mid V_\chi^{G^0_\mathfrak{l}}) = (-1)^{(\chi_m,1)}$ we are done.

5. Theorem F without tameness at l

In our special case $\mathbb{Q}(\zeta_n)^+/\mathbb{Q}$, our Theorem gives the formula (8.5) of [RW2] with an extra factor $\det(-1 \mid V_\chi^{G^0_\mathfrak{l}})$, but without assuming tameness of $l \neq 2$ in $\mathbb{Q}(\zeta_n)^+/\mathbb{Q}$. Based on (8.5), the remaining calculation proving Theorem F in [RW2] does not again invoke the tameness hypothesis, so we may just repeat the calculation including the extra factor. This yields the desired result in our special case.

It may now seem that the new and old (8.5) contradict each other: but in the tame case the factor $\chi \mapsto \det(-1 \mid V_\chi^{G^0_\mathfrak{l}})$ is ≈ 1 by Lemma 8.4. Indeed in [RW2], this was used at the end of the proof to eliminate the *extra* factor.

For a general extension K/k of totally real cyclotomic fields we choose an n so that $K \subset \mathbb{Q}(\zeta_n)^+$ and reduce to the special case $\mathbb{Q}(\zeta_n)^+/\mathbb{Q}$ in the following way. The assertion of Theorem F is that the class $\omega^{(l)}(K/k)$ in $K_0T(\mathbb{Z}_lG(K/k))$ is represented by $\chi \mapsto \chi(\Upsilon)\prod_{S_*^-}((\mathrm{N}\mathfrak{p}_k)^{-\dim V_\chi^{G^0_\mathfrak{p}}}$. Now (by [GRW, Theorem 3′])

$$\begin{array}{llll} \mathrm{defl}: K_0T(\mathbb{Z}_lG(\mathbb{Q}(\zeta_n)^+/\mathbb{Q})) \to K_0T(\mathbb{Z}_lG(K/\mathbb{Q})) & \text{takes} & \omega^{(l)}(\mathbb{Q}(\zeta_n)^+/\mathbb{Q}) \text{ to } \omega^{(l)}(K/\mathbb{Q}) \\ \text{and} \quad \mathrm{res}: K_0T(\mathbb{Z}_lG(K/\mathbb{Q})) \to K_0T(\mathbb{Z}_lG(K/k)) & \text{takes} & \omega^{(l)}(K/\mathbb{Q}) \text{ to } \omega^{(l)}(K/k)\,. \end{array}$$

The class $c(K/k) \in K_0T(\mathbb{Z}_lG(K/k))$ represented by $\chi \mapsto \prod_{S_*^-}(\mathrm{N}\mathfrak{p}_k)^{-\dim V_\chi^{G^0_\mathfrak{p}}}$ behaves in the same way: in view of [Fr, p.63] the proof of this is contained in the inflation/induction properties of Euler factors of Artin L-functions. So it remains to show that $b(K/k) \in K_0T(\mathbb{Z}_lG(K/k))$ represented by $\chi \mapsto \chi(\Upsilon)$ has the deflation/restriction properties: for these can then be applied to the relation $\omega^{(l)}(\mathbb{Q}(\zeta_n)^+/\mathbb{Q}) = b(\mathbb{Q}(\zeta_n)^+/\mathbb{Q}) + c(\mathbb{Q}(\zeta_n)^+/\mathbb{Q})$ of the special case to deduce the general one.

To discuss $b(K/k)$, let K_∞ be the cyclotomic l-extension of K and write $G = G(K/k)$, $G_\infty = G(K_\infty/k)$. There are "deflation" maps ρ (from [RW2, Appendix 4B])

$$\begin{array}{ccc} K_1(R^{-1}\mathbb{Z}_l[[G_\infty]]) & \stackrel{\partial}{\longrightarrow} & K_0R(\mathbb{Z}_l[[G_\infty]]) \\ \rho\downarrow & & \rho\downarrow \\ K_1(\mathbb{Q}_lG) & \stackrel{\partial}{\longrightarrow} & K_0T(\mathbb{Z}_lG) \end{array}$$

making the square above commute and, by definition of Υ ([RW2, Proposition 5.10]), it follows that $b(K/k)$ is the image of $\mho_S - \partial(\Theta_S)$ under $\rho : K_0R(\mathbb{Z}_l[[G_\infty]]) \to K_0T(\mathbb{Z}_lG)$. We will show, in §6, that the classes $\mho_S - \partial(\Theta_S)$, at the top of the Iwasawa tower, do have deflation and restriction. The desired properties of $b(K/k)$ then follow from the commutative squares below

$$\begin{array}{ccc} K_0R(\mathbb{Z}_l[[G_\infty]]) & \stackrel{\text{defl}}{\longrightarrow} & K_0R(\mathbb{Z}_l[[\overline{G}_\infty]]) \\ \rho\downarrow & & \overline{\rho}\downarrow \\ K_0T(\mathbb{Z}_lG) & \stackrel{\text{defl}}{\longrightarrow} & K_0T(\mathbb{Z}_l\overline{G}) \end{array} \qquad \begin{array}{ccc} K_0R(\mathbb{Z}_l[[G_\infty]]) & \stackrel{\text{res}}{\longrightarrow} & K_0R(\mathbb{Z}_l[[G'_\infty]]) \\ \rho\downarrow & & \rho'\downarrow \\ K_0T(\mathbb{Z}_lG) & \stackrel{\text{res}}{\longrightarrow} & K_0T(\mathbb{Z}_lG') \end{array}$$

compatible with the notation [17] of §6 (and verified via [RW2, Lemma 4B.1])

6. Appendix

This section is actually an addendum to [RW2, §4, §5.1] and the notation is taken from there [18]. We discuss how $\mho \in K_0R(\mathbb{Z}_lG_\infty)$ and $\Theta \in K_1(R^{-1}\mathbb{Z}_lG_\infty)$ behave with respect to deflation and restriction.

Recall the basic situation: l is a fixed odd prime number, k a totally real finite extension of $\mathbb{Q}$, k_∞ the cyclotomic l-extension of k and K_∞ a finite extension of k_∞ which is Galois, with group G_∞, over k.

Recall also that the elements $\mho$ and Θ depend only on a fixed sufficiently large finite set of primes of k, which contains all divisors of $l\infty$, and on a fixed generator γ_k of $\Gamma_k = G(k_\infty/k)$ ([RW2, pp.8,9]). Moreover, the definitions of $\mho$ and Θ in [RW2] require Leopoldt's conjecture and G_∞ abelian, respectively, which will be tacitly assumed at the appropriate places below.

Lemma A1. *Let $\overline{G}_\infty = G_\infty/N$ with N a finite normal subgroup of G_∞, and put $\overline{K}_\infty = {K_\infty}^N$. Then*

1. *defl: $K_0R(\mathbb{Z}_lG_\infty) \to K_0R(\mathbb{Z}_l\overline{G}_\infty)$ takes $\mho(K_\infty/k)$ to $\mho(\overline{K}_\infty/k)$,*
2. *defl: $K_1(R^{-1}\mathbb{Z}_lG_\infty) \to K_1(R^{-1}\mathbb{Z}_l\overline{G}_\infty)$ takes $\Theta(K_\infty/k)$ to $\Theta(\overline{K}_\infty/k)$.*

The first assertion is [RW2, Proposition 4.8] and the second follows, via the identification $K_1(R^{-1}\mathbb{Z}_lG_\infty) = (R^{-1}\mathbb{Z}_lG_\infty)^\times$ ([RW2, Lemma 5.5]), from [RW2, Proposition 5.4] which characterizes Θ already in $\mathcal{M} \cap (R^{-1}\mathbb{Z}_lG_\infty)^\times$ by $\chi(\Theta) = G_{\chi,S}(0)$ for all abelian l-adic characters χ of G_∞ with open kernel. Indeed, given such a character $\overline{\chi}$ of G_∞/N, then

$$\overline{\chi}(\overline{\Theta}) = (\text{infl}\,\overline{\chi})(\Theta) = G_{\text{infl}\,\overline{\chi}}(0) = G_{\overline{\chi}}(0)$$

with the last equality due to $L_l(s, \text{infl}\,\overline{\chi}) = L_l(s, \overline{\chi})$. Lemma A1 is proved.

[17] here $\overline{G}$ is intended to be any common factor group of G and $\overline{G}_\infty$ for which $\overline{\rho}$ is defined

[18] except for often omitting an index S

Lemma A2. *Let G'_∞ be an open subgroup of G_∞ and k' its fixed field. Let γ'_k be the unique generator of $\Gamma_{k'} = G(k'_\infty/k')$ which restricts to γ_k^m in Γ_k, where $m = [k' \cap k_\infty : k]$. Then*

1. $\mathrm{res} : K_0R(\mathbb{Z}_l G_\infty) \to K_0R(\mathbb{Z}_l G'_\infty)$ *takes* $\mho(K_\infty/k)$ *to* $\mho(K_\infty/k')$,
2. $\mathrm{res} : K_1(R^{-1}\mathbb{Z}_l G_\infty) \to K_1(R^{-1}\mathbb{Z}_l G'_\infty)$ *takes* $\Theta(K_\infty/k)$ *to* $\Theta(K_\infty/k')$,

with the set S' for K_∞/k' consisting of the primes of k' above those in the set S for K_∞/k.

For the proof we let H be the kernel of $G_\infty \to \Gamma_k$, observing that $H' = H \cap G'_\infty$ is then the kernel of $G'_\infty \to \Gamma_{k'}$. We should also point out that the multiplicative set R ([RW2, p.32]) is not as specific as it may seem: for if, in the notation there, $K/k \subset \tilde{K}/k$ are finite Galois inside K_∞/k, then taking the determinant of multiplication by $x \in \mathbb{Z}_l\Gamma_K$ on the free $\mathbb{Z}_l\Gamma_{\tilde{K}}$-module $\mathbb{Z}_l\Gamma_K$ gives $\mathrm{norm}(x) \in \mathbb{Z}_l\Gamma_{\tilde{K}}$ which has x as a factor in $\mathbb{Z}_l\Gamma_K$; inverting $\mathrm{norm}(x)$ thus inverts x. This allows us to take R inside $\mathbb{Z}_l G'_\infty$ below.

Three commutative diagrams are basic for the proof of 1.

$$
\begin{array}{ccccc}
\Delta G'_\infty & \rightarrowtail & \mathbb{Z}_l G'_\infty & \twoheadrightarrow & \mathbb{Z}_l \\
\downarrow & & \downarrow & & \| \\
\Delta G_\infty & \rightarrowtail & \mathbb{Z}_l G_\infty & \twoheadrightarrow & \mathbb{Z}_l \\
\downarrow & & \downarrow & & \\
F & = & F & &
\end{array}
\qquad
\begin{array}{ccccc}
X_\infty & \rightarrowtail & Y'_\infty & \twoheadrightarrow & \Delta G'_\infty \\
\| & & \downarrow & & \downarrow \\
X_\infty & \rightarrowtail & Y_\infty & \twoheadrightarrow & \Delta G_\infty \\
 & & \downarrow & & \downarrow \\
 & & F & = & F
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{Z}_l G'_\infty & \rightarrowtail & Y'_\infty & \twoheadrightarrow & \mathrm{coker}\,\Psi' \\
\downarrow & & \downarrow & & {\scriptstyle (1)}\downarrow \\
\mathbb{Z}_l G_\infty & \rightarrowtail & Y_\infty & \twoheadrightarrow & \mathrm{coker}\,\Psi \\
\downarrow & & \downarrow & & \downarrow \\
F & \overset{(2)}{\rightarrowtail} & F & \twoheadrightarrow & C
\end{array}
$$

All occurring modules are viewed as $\mathbb{Z}_l G'_\infty$-modules and all non-marked maps are the natural ones. The module F is, of course, free. The middle diagram arises from the translation functor in analogy to [RW2, (4.∞)]. The right diagram depends on a compatible choice of maps Ψ' and Ψ, as defined prior to [RW2, Proposition 4.5], to make the top left square commute. To construct these we choose a lift $\gamma' \in G'_\infty$ of $\gamma_{k'}$ and any $c'_\infty, d'_\infty, y'_\infty$ to form a Ψ'. We then choose a lift $\gamma \in G_\infty$ of γ_k, observing that then $\gamma' = \gamma^m h_0$ with $h_0 \in H$, and define

$$c_\infty = c'_\infty[(1 + \gamma + \cdots + \gamma^{m-1})e + (\gamma' - 1)(e' - e) + (1 - e')]$$

in the natural notation. From $e'e = e$, $h_0 e = e$ it follows that $d_\infty = c_\infty(\,(\gamma - 1)e + (1 - e)\,)$ is d'_∞, hence the image y_∞ of y'_∞ under $Y'_\infty \to Y_\infty$ in the middle diagram permits us to form a compatible Ψ.

This gives the top half of the right diagram, and then the full diagram by applying the snake lemma. In it, the arrow (1) is injective since its kernel, which is R-torsion ([RW2, Proposition 4.5]), embeds in the free $\mathbb{Z}_l G'_\infty$-module F. This also implies injectivity of arrow (2), which is easily checked, from the middle diagram, to be right multiplication by $d_\infty = d'_\infty$.

Now the bottom row of the right diagram shows $C \simeq F/Fd'_\infty$, hence

$$[C] = \partial'(F, d'_\infty) = \partial'(\mathbb{Z}_l G_\infty, d'_\infty) - \partial'(\mathbb{Z}_l G'_\infty, d'_\infty)$$

[19] by the left column. And the right column together with the definition of $\mho = \mho(K_\infty/k)$, $\mho' = \mho(K_\infty/k')$ gives

$$\mathrm{res}\,\mho - \mho' = \partial(\mathbb{Z}_l G'_\infty, c'_\infty) + [C] - \partial'(\mathbb{Z}_l G_\infty, c_\infty)\,.$$

[19] where the R^{-1} after $\partial'($ is suppressed

Substituting $[C]$, with $d'_\infty = d_\infty$ on $\mathbb{Z}_l G_\infty$, we get

$$\begin{aligned}
\operatorname{res}\mho - \mho' &= \partial'(\mathbb{Z}_l G'_\infty, c'_\infty) - \partial'(\mathbb{Z}_l G'_\infty, d'_\infty) + \partial'(\mathbb{Z}_l G_\infty, d_\infty) - \partial'(\mathbb{Z}_l G_\infty, c_\infty)\\
&= \partial'(\mathbb{Z}_l G_\infty, (\gamma-1)e + (1-e)) - \partial'(\mathbb{Z}_l G'_\infty, (\gamma'-1)e + (1-e'))\\
&= \partial'\Big(\operatorname{res}[\mathbb{Z}_l G_\infty, (\gamma-1)e + (1-e)] - [\mathbb{Z}_l G'_\infty, (\gamma'-1)e + (1-e')]\Big)\\
&= \partial' \mathrm{infl}\Big(\operatorname{res}[\mathbb{Z}_l \Gamma_k, \gamma_k - 1] - [\mathbb{Z}_l \Gamma_{k'}, \gamma_{k'} - 1]\Big)\,,
\end{aligned}$$

because of the commutative square

$$\begin{array}{ccc}
K_1(R^{-1}\mathbb{Z}_l G_\infty) & \xrightarrow{\text{res}} & K_1(R^{-1}\mathbb{Z}_l G'_\infty)\\
\mathrm{infl}\uparrow & & \mathrm{infl}\uparrow\\
K_1(R^{-1}\mathbb{Z}_l \Gamma_k) & \xrightarrow{\text{res}} & K_1(R^{-1}\mathbb{Z}_l \Gamma_{k'})
\end{array}$$

in which the lower restriction in the diagram comes from the inclusion $\Gamma_{k'} \to \Gamma_k$ which is compatible with $G'_\infty \hookrightarrow G_\infty$, and therefore takes $\gamma_{k'}$ to γ_k^m.

To compare $[\mathbb{Z}_l\Gamma_{k'}, \gamma_{k'}-1]$ and $\operatorname{res}[\mathbb{Z}_l\Gamma_k, \gamma_k - 1]$ we use the isomorphism $\det : K_1(R^{-1}\mathbb{Z}_l\Gamma_{k'}) \to (R^{-1}\mathbb{Z}_l\Gamma_{k'})^\times$ ([RW2, Lemma 5.5]) which sends the first to $\gamma_{k'} - 1$ and the second to the determinant of the $R^{-1}\mathbb{Z}_l\Gamma_{k'}$-endomorphism $\gamma_k - 1$ of $R^{-1}\mathbb{Z}_l\Gamma_k$: using the basis $1, (\gamma_k - 1), \cdots, (\gamma_k-1)^{m-1}$ the relation $(\gamma_k-1)^{m-1}\cdot(\gamma_k - 1) = (\gamma_k^m - 1)\cdot 1 - \sum_{i=1}^{m-1}\binom{m}{i}(\gamma_k-1)^i$ implies that this determinant is $(-1)^{m-1}(\gamma_{k'} - 1)$. These agree since m, which is a power of l, is odd.

We now turn to 2. of Lemma A2. The map $\operatorname{res}^{G_\infty}_{G'_\infty}$ takes $\Theta \in K_1(R^{-1}\mathbb{Z}_l G_\infty) = (R^{-1}\mathbb{Z}_l G_\infty)^\times$ to the determinant of the $R^{-1}\mathbb{Z}_l G'_\infty$-endomorphism $x \mapsto x\Theta$ of $R^{-1}\mathbb{Z}_l G_\infty$. Now $l^r\Theta \in \mathbb{Z}_l G_\infty$ for some r (by $\Theta \in \mathcal{M}$ as in the proof of Lemma A1) and $\mathbb{Z}_l G_\infty$ free over $\mathbb{Z}_l G'_\infty$ implies that the determinant of $l^r\Theta$ is in $\mathbb{Z}_l G'_\infty$, hence $l^{rm}\operatorname{res}\Theta \in \mathbb{Z}_l G'_\infty$. It follows from the second paragraph of the proof of [RW2, Proposition 5.4] that $\operatorname{res}\Theta$ is characterized by the values $\chi'(\operatorname{res}\Theta)$ with χ' varying over l-adic characters of G'_∞ with open kernel.

Every such χ' factors through some finite quotient G' of G'_∞ and $\rho'\operatorname{res}\Theta$ has representing homomorphism

$$\chi' \mapsto \chi'(\operatorname{res}\Theta) = (\operatorname{ind}_{G'}^G \chi')(\rho\Theta) = \prod_{\chi|\chi'} \chi'(\Theta)$$

$$\begin{array}{ccc}
K_1(R^{-1}\mathbb{Z}_l G_\infty) & \xrightarrow{\text{res}} & K_1(R^{-1}\mathbb{Z}_l G'_\infty)\\
\rho\downarrow & & \rho'\downarrow\\
K_1(\mathbb{Q}_l G) & \xrightarrow{\text{res}} & K_1(\mathbb{Q}_l G')
\end{array}$$

([Fr, p.63]), where χ runs through all abelian characters of G_∞ extending χ'.

Thus $\chi'(\operatorname{res}\Theta) = \prod_{\chi|\chi'}\chi(\Theta)$ for all χ', and $\operatorname{res}\Theta$ must agree with $\Theta' = \Theta(K_\infty/k')$ if we can show that $\chi'(\Theta') = \prod_{\chi|\chi'}\chi(\Theta)$ for all χ'. This follows from the induction property $\prod_{\chi|\chi'} L_{l,S}(1-s,\chi) = L_{l,S'}(1-s,\chi')$ of (truncated) l-adic L-functions. Namely, substituting the Deligne-Ribet power series into this (compare [RW2] after the proof of Proposition 5.3) and noting that $\gamma_{k'}$ acts by u^m if γ_k acts by u on ζ_{l^∞} we get

$$(*)\qquad \frac{\prod_{\chi|\chi'} G_{\chi,S}(u^s-1)}{G_{\chi',S'}(u^{ms}-1)} = \frac{\prod_{\chi|\chi'} H_\chi(u^s-1)}{H_{\chi'}(u^{ms}-1)} = \begin{cases} (-1)^{m-1} & \text{if } \chi'_{|H'} = 1\\ 1 & \text{if } \chi'_{|H'} \neq 1\,. \end{cases}$$

For $\chi'_{|H'} \neq 1$ implies $\chi_{|H} \neq 1$ for all $\chi|\chi'$; and if $\chi'_{|H'} = 1$ then there are m characters $\chi|\chi'$ with $\chi_{|H} = 1$, which can be numbered $\chi_0, \cdots, \chi_{m-1}$ with $\chi_i(\gamma_k) = \chi_1(\gamma_k)\zeta^i$, with ζ a primitive m-th root of unity. Then $\frac{\prod_{\chi|\chi'} H_\chi(u^s-1)}{H_{\chi'}(u^{ms}-1)} = \frac{\prod_{i=1}^m(\chi_0(\gamma_k)\zeta^i u^s - 1)}{\chi'(\gamma_{k'})u^{ms}-1} = (-1)^{m-1}\frac{\chi_0(\gamma_k^m)u^{ms}-1}{\chi'(\gamma_{k'})u^{ms}-1} = (-1)^{m-1}$. Since m is odd, substituting $s=0$ gives $\frac{\prod_{\chi|\chi'}\chi(\Theta)}{\chi'(\Theta')} = 1$ and Lemma A2 is proved.

Remark. When $l = 2$, $\mho - \partial\Theta$ still has deflation and restriction and an easy modification of $\mho, \Theta$ would also recover Lemma A1 and A2.

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